Firm Scope and Innovation: The Role of Intangibles*

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Abstract

This paper develops a unified theoretical framework to study how the composition of intangible investment and innovation direction jointly shape long-run growth and creative destruction. I build an endogenous growth model in which firms choose between vertical and horizontal innovation and allocate investment across R&D and firm-specific intangibles (brand value and organizational capital). Using firm-level data, I discipline the model and find that productivity, markups, and intangible investment composition vary systematically with firm scope: firms with narrower scope concentrate investment in knowledge capital, whereas firms with broader scope allocate relatively more to brand value and organizational capital. The calibrated model shows that firm scope determines intangible composition, which governs innovation direction, and that growing scope and organizational capital jointly raise entry barriers and suppress creative destruction. Size-based policies misrepresent firms' investment incentives by treating vertical and horizontal expansion symmetrically, whereas scope-dependent policies targeting intangible-driven externalities lead to higher rates of creative destruction and innovation.

Keywords: Schumpeterian growth, step-by-step innovation, intangibles, firm dynamics, span of control.

JEL Classification: E22, O31, O32, O33, O34.

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1. Introduction

Horizontal and vertical innovation are two core strategies in the modern endogenous growth literature for generating long-run growth through creative destruction. Firms can engage in horizontal innovation by expanding their product portfolios (Klette and Kortum, 2004) or pursue vertical innovation by improving the quality of existing products (Aghion, Harris, Howitt, and Vickers, 2001). At the same time, literature also documents that firms invest in intangibles not only to innovate but also to build intangible assets such as brand value and organizational capital that strengthen market positions rather than drive innovation-led growth (Cavenaile and Roldan-Blanco, 2021; Crouzet and Eberly, 2019).

While the directions of innovation and composition of intangibles are well documented separately, their interaction remains largely unexplored. To be concrete, Apple is a vertically integrated firm focused on mobile communication technologies and invests heavily in R&D. By contrast, 3M is active in transportation, safety, consumer, and health care industries, and although R&D remains its largest intangible investment, it allocates a substantially larger share toward brand and organizational capital relative to Apple[1]. This contrast illustrates why firm size is an insufficient basis for innovation policy: a tax or subsidy that treats Apple and 3M symmetrically penalizes knowledge spillovers and rewards market-positioning intangibles equally, regardless of their aggregate consequences. Yet existing theories provide no framework for distinguishing these two firms' investment incentives or for designing policies that account for this heterogeneity.

What are the aggregate consequences of the interaction between innovation direction and intangible composition for economic growth and creative destruction? To answer this question, I proceed in two steps. First, I develop a unified theoretical framework in which firms engage in both vertical and horizontal innovation and allocate investment across different types of intangibles. Second, I document a new set of empirical regularities showing how firm-level outcomes and investment choices vary systematically with

[1] The measurement details are given in Section 3.2.

firm scope, which I use to discipline the model. The calibrated model shows that firms operating a narrower scope concentrate investment in knowledge capital and tend toward vertical innovation, whereas firms with a broader scope allocate relatively more to brand value and organizational capital, and tend toward horizontal innovation. This joint determination of innovation direction and intangible composition implies that policies conditioned solely on firm size are insufficient, as firm size contains vertical and horizontal dimensions and therefore misrepresents firms' intangible investment incentives. By contrast, scope-dependent policies that target the externalities associated with intangible heterogeneity lead to higher rates of creative destruction and innovation.

To formalize intangible heterogeneity, I classify intangible assets according to their spillover effects across firms. In this paper, transferable intangibles consist of knowledge capital[2] with the associated R&D investments. These assets can be transferred between firms and generate spillover effects due to their non-rivalrous and partially excludable nature (Romer, 1990). In contrast, embedded intangibles comprise brand value[3] and organizational capital[4] are inherently firm-specific and inseparable from the firm that created them and do not generate spillovers. Consequently, when a firm exits the market, the economic value of embedded intangibles becomes a sunk cost.

In Section 2, I develop an endogenous growth model that the economy consists of a single final-good sector and a continuum of intermediate-good sectors. In each intermediate goods sector, a single superstar firm and a continuum of fringe firms produce intermediate goods for the final goods sector with a la Bertrand competition. Superstar firms can invest in embedded intangibles, which improve managerial productivity and

[2]Following Griliches (1979), R&D investment generates a stock of knowledge capital that accumulates over time. Examples of such knowledge include patents and innovation-related software, which illustrate the types of assets that embody technological know-how.

[3]Brand value is a demand shifter, positively influencing the perceived quality of a firm's output (Cavenaile and Roldan-Blanco, 2021; Cavenaile, Celik, Roldan-Blanco, and Tian, 2025). Evidence also suggests that brand value targeted marketing by increasing consumer awareness, thereby incentivizing substantial firm investment in advertising (Cavenaile, Celik, Perla, and Roldan-Blanco, 2025; Baslandze, Greenwood, Marto, and Moreira, 2023)

[4]Organizational capital conceptualized as managerial productivity, including the firm's embodied managerial talent and its contribution to future production profitability (Carlin, Chowdhry, and Garmaise, 2012; Eisfeldt and Papanikolaou, 2013; Prescott and Visscher, 1980)

the perceived quality of their products through brand value and organizational capital. Alternatively, they can invest in transferable intangibles, which enhance product quality in two ways: (i) by vertically improving existing products and (ii) by expanding their scope through entry into new sectors. Fringe firms produce homogeneous products and cannot accumulate brand value and organizational capital. When a superstar firm exits a sector, its transferable intangibles—the established product quality in that sector—become freely available to fringe firms. For fringe firms, the only path to becoming a superstar is through radical innovation.

The model incorporates two key frictions. First, the two types of intangibles differ in their scalability: embedded intangibles generate economies of scope as their benefits accumulate across sectors without proportional increases in cost, whereas transferable intangibles require separate investment in each sector. Second, while vertical innovation imposes no span-of-control constraints, expanding scope divides managerial attention across sectors, therefore reducing managerial quality per sector (Lucas, 1978)[5].However, as a superstar firm expands into more sectors, it accumulates a larger stock of organizational capital and managerial talent across its entire portfolio, so that while managerial quality per sector declines, the superstar's total managerial capacity increases with scope. When facing a competitive threat, the incumbent superstar can temporarily redeploy this accumulated total capacity onto a single contested sector, making the entry barrier it poses to challengers increasing in both firm scope and organizational capital.

These two frictions jointly generate three predictions. (i) Embedded intangibles are a prerequisite for horizontal innovation: entry into a new sector is costly unless a superstar firm has accumulated sufficient brand value and organizational capital to offset the costs imposed by span-of-control frictions. (ii) Once this threshold is reached, further expansion becomes self-reinforcing: broader scope raises the return to embedded intangibles through economies of scope, inducing greater investment in brand value and organizational capital, which in turn lowers the cost of entering additional sectors. (iii)

[5] See also Jovanovic (2025). Acemoglu, Akcigit, Alp, Bloom, and Kerr (2018) models a related trade-off in which skilled labor allocated to operational activities creates a cost of expanding scope, though without a direct effect on efficiency or pricing power.

As scope and embedded intangible stock grow jointly, the superstar's total managerial capacity increases, raising the threshold that entrants must clear to displace the incumbent and thereby suppressing creative destruction. Taken together, these predictions generate a wedge between private and social returns: superstar firms with broader scope investing predominantly in embedded intangibles erect larger entry barriers and generate smaller knowledge spillovers than firms with narrower scope concentrating investment in knowledge capital.

In Section 3, I combine firm-level data from Compustat with the product-market fluidity dataset of Hoberg, Phillips, and Prabhala (2014) to document how firm-level outcomes vary systematically with firm scope. As firm scope rises, markups, productivity, the transferable-to-embedded intangible ratio, and competitive pressure all decline. I further investigate the heterogeneity in these patterns by exploiting variation in patent value (Kogan, Papanikolaou, Seru, and Stoffman, 2017) as an innovation shock in local projections, comparing the responses of single-sector and multi-sector firms. Single-sector firms exhibit stronger reallocation toward transferable intangibles, larger productivity gains, higher markup responses, and greater competitive entry following an equivalent shock, while multi-sector firms display consistently weaker responses across all dimensions, consistent with the model's prediction.

In Section 4, I discipline the model using the simulated method of moments, calibrating parameters directly from firm-level moments in the data. To isolate the role of each friction, I calibrate two counterfactual economies, one without span-of-control constraints and one without embedded intangibles. Removing span-of-control constraints allows firms to operate across more sectors, raising markups and aggregate growth. Eliminating embedded intangibles concentrates production on a single sector, lowers markups modestly, and raises aggregate growth through a substantial increase in creative destruction. Together, these counterfactuals confirm that both frictions are necessary: span-of-control constraints govern the cost of horizontal expansion, while embedded intangibles govern the entry barriers that suppress creative destruction, and neither friction alone is sufficient to match the empirical regularities documented in the data.

In Section 6, I use the calibrated model to examine the aggregate implications of intangible heterogeneity for innovation policy. I quantify misallocation through two channels: markup dispersion, which accounts for only 0.5% of output loss, and entry barriers created by the joint expansion of scope and embedded intangibles, whose removal more than triples aggregate output through reallocation toward transferable intangible investment and quality improvement. This asymmetry reveals that the primary source of misallocation is not pricing distortions but the suppression of creative destruction by scope-driven entry barriers. I evaluate three policy experiments, a size-dependent profit tax, a tax on embedded intangible investment, and a tax on expansion investment, each combined with three redistribution schemes: (i) lump-sum transfers to consumers, (ii) uniform subsidies to all fringe firms, and (iii) targeted subsidies to fringe firms whose incumbent competitors are narrow-scope superstar firms that have not yet accumulated sufficient embedded intangibles. The third redistribution scheme consistently delivers the largest welfare gains across all tax instruments, as it directs resources toward the fringe firms with the lowest entry barriers and the highest potential for radical innovation, thereby accelerating creative destruction and amplifying the aggregate welfare gain. These results establish that scope-dependent policies targeting the externalities associated with intangible heterogeneity, combined with redistribution toward the margin of creative destruction, are considerably more effective in restoring long-run growth than instruments conditioned on firm size alone.

Related Literature. A growing body of work documents that intangible investment increases market concentration and markups (Chiavari and Goraya, 2025; Crouzet and Eberly, 2019; Weiss, 2020). De Ridder (2024) treats software as a firm-specific fixed cost and shows that firms with higher software capability strategically increase their fixed-cost investment to lower their marginal production costs, which creates entry barriers. Aghion, Bergeaud, Boppart, Klenow, and Li (2023) distinguishes between product and process innovation and documents the role of information and communication technologies in driving concentration. In both frameworks, intangible capabilities are treated as a firm's inborn characteristics rather than endogenous investment choices. Cavenaile and Roldan-

Blanco (2021) and Cavenaile, Celik, Roldan-Blanco, and Tian (2025) show that advertising can substitute for R&D and dampen innovation intensity, while Pearce and Wu (2025) examines brand-value transfer in the context of market concentration. I contribute to this literature by establishing intangible composition as an endogenous determinant of innovation direction, showing that how firms allocate investment across R&D, brand value, and organizational capital governs their choice between vertical and horizontal innovation and shapes the dynamics of market competition.

The literature has predominantly explained firms' choice of innovation direction through differences in firm size. Akcigit and Kerr (2018) show that smaller firms are more likely to engage in external innovation while larger firms tend toward quality-improving vertical innovation. Berlingieri, De Ridder, Lashkari, and Rigo (2025) document that firms frequently expand through sequential product diversification rather than by first improving existing products, while Garcia-Macia, Hsieh, and Klenow (2019) find that most within-firm innovation arises from incumbents improving existing products rather than entering new lines. This paper establishes intangible composition as a determinant of innovation direction that operates independently of firm size. I show that firms of similar size optimally choose different innovation paths depending on their allocation of investment across transferable and embedded intangible assets, which firm size alone cannot account for the observed heterogeneity in innovation strategies.

The joint expansion of firm scope and embedded intangible investment also provides a complementary mechanism underlying several documented secular trends: declining knowledge spillovers (Akcigit and Ates, 2021; Akcigit and Ates, 2023), deteriorating patent quality (Olmstead-Rumsey, 2019), production lock-in (Casal, 2024), and rising entry barriers (Gutiérrez and Philippon, 2019). As firm scope expands, investment shifts from transferable toward embedded intangibles, depressing knowledge spillovers and raising entry barriers through the accumulation of organizational capital and scope.

This paper also contributes to the misallocation literature (Hsieh and Klenow, 2009;Restuccia and Rogerson, 2008) by identifying two distinct channels through which intangible heterogeneity generates allocative distortions. The first operates through markup dispersion (Peters, 2020; Edmond, Midrigan, and Xu, 2023), driven by the accumulation of both transferable and embedded intangibles. The second operates through entry barriers: the joint expansion of embedded intangibles and firm scope makes creative destruction asymmetric, generating persistent misallocation that markup-based measures substantially understate.

Finally, the literature on span-of-control has focused primarily on hierarchical organization, knowledge-flow frictions, and managerial ability (Smeets, Waldman, and Warzynski, 2019; Bandiera, Prat, Sadun, and Wulf, 2014; Garicano, 2000; Bloom and Van Reenen, 2007) and with (Jovanovic, 2025) linking span-of-control constraints to the firm-size distribution at the macro level. I contribute to this literature by disaggregating firm size into vertical and horizontal dimensions of innovation and showing that span-of-control frictions arise only from horizontal expansion. This distinction isolates how scope-driven managerial constraints shape intangible investment, innovation direction, and aggregate growth.

2. Theoretical Model

The theoretical framework builds on the step-by-step vertical innovation model of Akcigit and Ates (2023) and the vertical and horizontal innovation model of Peters (2020). I combine both frameworks by introducing two distinct types of intangible investment and span-of-control frictions that arise exclusively from expanding scope.

2.1. Economic Environment

Preferences. Time is continuous and indexed by t. Household preferences are described by a logarithmic utility function

$$\int_0^\infty e^{-\rho t} \ln(C_t)\, dt, \tag{1}$$

where C_t denotes household consumption and $\rho > 0$ is the time discount rate. The household budget constraint is

$$\dot{A}_t = r_t A_t + w_t - C_t, \tag{2}$$

where A_t denotes total household assets, w_t is the wage rate, and r_t is the interest rate. Labor is supplied inelastically, and the price of the consumption good is normalized to 1, so that w_t and r_t are expressed in units of the consumption good. Since households own all firms in the economy, total assets equal the aggregate value of superstar and fringe firms across all intermediate-good sectors

$$A_t = \int_0^1 \left(V_{sjt} + V_{fjt}\right) dj,$$

where V_{sjt} and V_{fjt} denote the values of the superstar and fringe firm in sector j at time t.

Final Good Technology and Market Structure. The final good is produced by aggregating a continuum of intermediate varieties $j \in [0, 1]$ according to

$$\ln(Y_t) = \int_0^1 \ln(y_{jt})\, dj, \tag{3}$$

in a perfectly competitive market. In each intermediate-good sector, a single superstar firm and a continuum of fringe firms compete à la Bertrand competition to supply the final good producer. Output in each sector is aggregated with a constant elasticity of substitution technology

$$y_{jt} = \left(\underbrace{\frac{\chi(e_{st})}{1 + \chi(e_{st})}}_{\substack{\Xi(e_{st}) \\ \text{Perceived Quality}}} y_{sjt}^{\varepsilon} + \underbrace{\frac{1}{1 + \chi(e_{st})}}_{\frac{\Xi(e_{st})}{\chi(e_{st})}} y_{fjt}^{\varepsilon} \right)^{\frac{1}{\varepsilon}}, \tag{4}$$

where $\varepsilon \in (0, 1)$. A superstar firm s may operate across multiple sectors simultaneously, characterized by the set of sectors for which it holds the leading technology, $J_s \subseteq [0, 1]$, with $n_s = |J_s| \in \mathbb{Z}_+$ denoting the number of such sectors.

In each sector, the superstar firm produces a differentiated good whose perceived quality is increasing in its brand value. Specifically, $\chi(\xi e_{st}) = (\xi e_{st})^{\beta}$ is an endogenous and concave demand shifter, where e_{st} denotes the embedded intangibles of superstar firm s, $\xi \in (0,1)$ is the share of embedded intangibles associated with brand value, and $\beta \in (0,1)$ governs the curvature of the demand shifter. The term $\Xi(e_{st})$ captures the notion that a higher stock of brand value raises the perceived quality of the superstar's product relative to the fringe. For simplicity, the brand value of fringe firms is normalized to one. Fringe firms produce a homogeneous good and are represented by the aggregate

$$y_{fjt} = \int_0^1 y_{ijt}\, di, \tag{5}$$

where each fringe firm $i \in (0,1)$ takes the market price as given.

Superstar Firm Production. The production function for superstar firm s in sector j at time t is

$$y_{sjt} = \underbrace{q_{sjt}}_{\text{Product Quality}} \underbrace{\psi(e_{st}, n_{st})}_{\text{Managerial Quality}} \underbrace{l_{sjt}}_{\text{Labor Input}}, \tag{6}$$

where q_{sjt} denotes product quality, l_{sjt} is labor input, and $\psi(e_{st}, n_{st})$ captures managerial productivity, defined as

$$\psi(e_{st}, n_{st}) = \frac{\big((1-\xi)\, e_{st}\big)^{\alpha}}{\gamma\, n_{st}^{\alpha}}.$$

The component $(1-\xi)e_{st}$ represents the share of embedded intangibles allocated to organizational capital, which governs managerial efficiency. The variable n_{st} denotes the number of sectors operated by firm s at time t. As firm scope n_{st} increases, managerial productivity per sector declines due to the span-of-control friction: the firm's managerial attention is distributed across a larger number of sectors, reducing the managerial capacity allocated to each individual sector. The parameter α governs the curvature of both organizational capital and the span-of-control constraint, while γ is a scale parameter. Aggregating superstar firm s managerial productivity across all active sectors generates total managerial capacity[6]

$$\Lambda \cdot e_{st}^{\alpha} \cdot n_{st}^{1-\alpha}, \quad \text{with} \quad \Lambda \equiv \frac{(1-\xi)^{\alpha}}{\gamma}.$$

When $0 < \alpha < 1$, total managerial capacity follows a Cobb-Douglas form, exhibiting diminishing marginal returns to both organizational capital and scope.[7] Total managerial capacity governs the incumbent superstar's ability to defend its market position against challenger entry, a mechanism formalized in the creative destruction part.

Fringe Firm Production. Fringe firms produce output using a linear technology:

$$y_{fjt} = q_{fjt}\, l_{fjt}, \tag{7}$$

where q_{fjt} denotes productivity and l_{fjt} is labor input. Fringe firms operate in a single sector, cannot accumulate embedded intangibles, and have managerial quality normalized to one. Their productivity is inherited: when a superstar firm exits a sector, its transferable intangibles become publicly available, and fringe firms adopt the previous superstar's productivity level.

Investment and Innovation. Superstar firms pursue three distinct innovation strategies. They can invest in transferable intangibles to either improve the quality of an existing product through vertical innovation or expand into a new sector through horizontal innovation. Additionally, they can invest in embedded intangibles to raise their brand value and organizational capital across all sectors in which they operate (see in Figure 1). The first two strategies require sector-specific investment because product quality improvements embody knowledge specific to each sector's technology. By contrast, embedded intangible investment is firm-specific: an improvement in brand value or organizational capital simultaneously raises the embedded intangible stock of all sectors within the firm's portfolio, reflecting the economies of scope that embedded intangibles generate.

[6]Managerial productivity is symmetric across sectors, so total managerial capacity equals per-sector managerial productivity multiplied by the number of active sectors.

[7]When $\alpha > 1$, diseconomies of scope emerge: coordination costs outweigh the benefits of expansion and total managerial capacity decreases with firm scope.

The variables I_{sjt}^{Emb}, I_{sjt}^{Ver}, and I_{sjt}^{Hor} denote the investment of superstar firm s in embedded intangibles, vertical innovation in an existing sector, and horizontal innovation into a new sector, respectively. Each unit of investment generates a successful flow rate of innovation: z_{jt}^{Emb} for embedded intangible, z_{jt}^{Ver} for vertical, and z_{jt}^{Hor} for horizontal innovation investment. Investments are subject to convex costs:

$$I_{jt}^{\text{Ver}} = \gamma^{\text{Ver}} \left(z_{sjt}^{\text{Ver}}\right)^{\vartheta^{\text{Ver}}} Y_t, \qquad I_{jt}^{\text{Hor}} = \gamma^{\text{Hor}} \left(z_{jt}^{\text{Hor}}\right)^{\vartheta^{\text{Hor}}} Y_t, \tag{8}$$

$$\text{and} \qquad I_{jt}^{\text{Emb}} = \gamma^{\text{Emb}} \left(z_{jt}^{\text{Emb}}\right)^{\vartheta^{\text{Emb}}} Y_t,$$

where the cost parameters γ^{Ver}, γ^{Hor}, and γ^{Emb} govern the scale of each investment cost function, ϑ^{Ver}, ϑ^{Hor}, and ϑ^{Emb} govern their curvature, and all costs scale with aggregate output Y_t. The total investment in transferable intangibles by a superstar s in sector j at time t is

$$I_{sjt}^{\text{T}} = I_{sjt}^{\text{Ver}} + I_{sjt}^{\text{Hor}}. \tag{9}$$

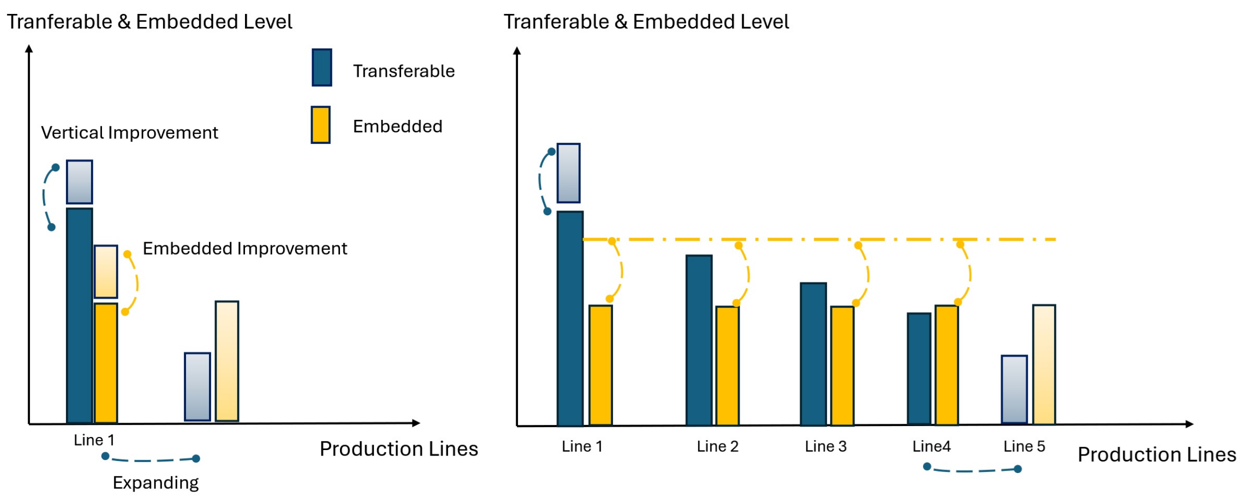

Figure 1. Firm Investment and Innovation Types

Fringe firms invest only in transferable intangibles within their sector, targeting radical innovations that would allow them to displace the incumbent superstar. Their investment cost function takes the form

$$I_{fjt} = \gamma^f \left(z_{fjt}\right)^{\vartheta f} Y_t, \tag{10}$$

where γ^f is the cost scale parameter and ϑ^f governs the curvature of investment.

A successful innovation — whether vertical within an existing sector or horizontal into a new sector — improves product quality by a factor of $\lambda > 1$. A successful embedded intangible investment raises brand value and organizational capital by a factor of $\theta > 1$ across all active sectors simultaneously. Product quality and embedded intangibles therefore evolve as

$$q_{sjt} = \lambda^{m_{sjt}} q_{sj0}, \quad \text{and} \quad e_{st} = \theta^{k_{st}} e_{s0}, \tag{11}$$

where m_{sjt} and k_{st} denote the cumulative number of product-quality and embedded intangible innovations by firm s in sector j up to time t, respectively, and initial levels are normalized to $q_{sj0} = 1$ and $e_{s0} = 1$.

Upon the exit of a superstar firm, its transferable technology becomes freely available to fringe firms in sector j. The quality gap between the superstar and the fringe in sector j at time t is therefore

$$\frac{q_{sjt}}{q_{fjt}} = \lambda^{m_{sjt} - m_{fjt}} = \lambda^{m_{jt}}, \tag{12}$$

where $m_{jt} \equiv m_{sjt} - m_{fjt}$ denotes the transferable intangible gap. Since the embedded intangible stock of fringe firms is normalized to one, the gap in brand value and organizational capital is given by the superstar's embedded stock:

$$\frac{e_{st}}{e_{ft}} = \frac{\theta^{k_{st}}}{1} = \theta^{k_{st}}, \tag{13}$$

where k_{st} denotes the embedded intangible gap. To ensure a finite state space, I impose upper bounds $\bar{m}$ and $\bar{k}$ on the quality gap m_{jt} and the embedded gap k_{st}, respectively [8].

Both superstar and fringe firms can displace the incumbent superstar through creative destruction, but their paths to entry differ fundamentally due to asymmetric capabilities in innovation and intangible accumulation.

[8]The span of control constraint imposes a natural upper bound $\bar{n}$ on expanding scope.

Assumption (Managerial Reallocation). *When an incumbent superstar faces competition from a challenger, it can temporarily concentrate its entire managerial capacity on the contested sector. This reallocation is costless in the short run and determines the outcome of the price competition stage.*

Superstar Creative Destruction. A superstar firm expanding its scope innovates in a randomly selected sector j', improving product quality by a factor of λ such that $q_{sjt} = \lambda\, q_{s'jt}$, and displaces the incumbent if its marginal cost falls below the incumbent superstar:

$$\frac{w_t}{\Lambda\, q_{sjt}\, e_{st}^{\alpha}\, n_{st}^{1-\alpha}} < \frac{w_t}{\Lambda\, q_{s'jt}\, e_{s't}^{\alpha}\, n_{s't}^{1-\alpha}}.$$

The probability that the entrant successfully displaces the incumbent is

$$\mathbb{P}_{s>s'} \;\equiv\; \sum_{m_{s'}=1}^{\bar{m}} \sum_{k_{s'}=1}^{\bar{k}} \sum_{n_{s'}=1}^{\bar{n}} \mathbb{I}\big\{\lambda\, (\theta^{k_s})^{\alpha}\, n_{st}^{1-\alpha} > (\theta^{k_{s'}})^{\alpha}\, n_{s't}^{1-\alpha}\big\}\, \mu_t(m_{s'}, k_{s'}, n_{s'}),$$

where the left-hand side captures the entrant's total managerial capacity after the quality improvement and the right-hand side captures the incumbent's total managerial capacity under temporary reallocation. The term $\mu_t(m_{s'}, k_{s'}, n_{s'}) \in [0,1]$ denotes the mass of superstar firms in state $(m_{s'}, k_{s'}, n_{s'})$ at time t, defined over the finite and discrete state space $\mathcal{M} \times \mathcal{K} \times \mathcal{N}$, with $\mathcal{M} = \{1, \ldots, \bar{m}\}$, $\mathcal{K} = \{1, \ldots, \bar{k}\}$, and $\mathcal{N} = \{1, \ldots, \bar{n}\}$, satisfying

$$\sum_{m_{s'}, k_{s'}, n_{s'}} \mu_t(m_{s'}, k_{s'}, n_{s'}) = 1. \tag{14}$$

Fringe Firm Creative Destruction. Fringe firms cannot accumulate embedded intangibles and therefore cannot compete with the incumbent on managerial capacity. Their only path to superstar status is through radical innovation, which requires jumping multiple steps on the quality ladder by a factor $\lambda^{\bar{m}} > \lambda$, such that $q_{sjt} = \lambda^{\bar{m}}\, q_{s'jt}$.[9] Since fringe firms are symmetric and invest identically within each sector, a fringe firm successfully

[9] Fringe firms in the model correspond to small firms in the data. A vast literature documents that small and young firms grow faster than their larger counterparts (Evans, 1987 Haltiwanger, Jarmin, and Miranda, 2013).

displaces the incumbent in sector j if and only if

$$\mathbb{I}_{f>s'} \equiv \mathbb{I}\left\{\lambda^{\bar{m}} > (\theta^{k_{s'}})^{\alpha} n_{s'}^{1-\alpha}\right\},$$

where the right-hand side depends solely on the incumbent's state $(k_{s'}, n_{s'})$. This condition, together with the challenger superstar entry condition, establishes that entry barriers are increasing in both the incumbent's embedded intangible stock $\theta^{k_{s'}}$ and its scope $n_{s'}$, regardless of whether the challenger is a fringe firm or a rival superstar.

2.2. Equilibrium

This section characterizes the general equilibrium of the model, which consists of a static and a dynamic component. The analysis begins with the static equilibrium, determining prices and allocations for a given set of states. Subsequently, I define the Markov Perfect Equilibrium for the dynamic game, outlining the value functions, optimal policy functions, and the evolution of the aggregate state distribution.

Household's Problem. A household maximizes utility subject to the budget constraint, yielding the Euler equation:

$$\frac{\dot{C}_t}{C_t} = r_t - \rho. \tag{15}$$

Along the balanced growth path, consumption and output grow at the same rate $g = r - \rho$, and the transversality condition holds.

Final and Intermediate Good Sectors. The final-good producer's demand for intermediate goods in sector j satisfies

$$p_{jt} = \frac{Y_t}{y_{jt}}. \tag{16}$$

This implies the following demand functions for the superstar and fringe firms,

$$y_{sjt} = p_{jt}^{\frac{\varepsilon}{1-\varepsilon}} p_{sjt}^{\frac{1}{\varepsilon-1}} Y_t \big(\Xi(e_s)\big)^{\frac{1}{1-\varepsilon}} \quad \text{and} \quad y_{fjt} = p_{jt}^{\frac{\varepsilon}{1-\varepsilon}} p_{fjt}^{\frac{1}{\varepsilon-1}} Y_t \big(\Xi(e_s)/\chi(e_s)\big)^{\frac{1}{1-\varepsilon}}, \tag{17}$$

where p_{sjt} and p_{fjt} are the prices charged by the superstar and fringe firms, respectively, and p_{jt} is the ideal price index for sector j:

$$p_{jt} = \left(\left(\Xi(e_s)\right)^{\frac{-1}{\varepsilon-1}} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} + \left(\Xi(e_s)/\chi(e_s)\right)^{\frac{-1}{\varepsilon-1}} p_{fjt}^{\frac{\varepsilon}{\varepsilon-1}} \right)^{\frac{\varepsilon-1}{\varepsilon}}. \tag{18}$$

Prices and Market Share. The Cobb-Douglas aggregator implies equal expenditure shares across all sectors. The market share of superstar firm s in sector j at time t is

$$\frac{p_{sjt} y_{sjt}}{p_{jt} y_{jt}} = \frac{p_{sjt} y_{sjt}}{Y_t} = p_{jt}^{\frac{\varepsilon}{1-\varepsilon}} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} \, \Xi(e_s)^{\frac{1}{1-\varepsilon}} \equiv \phi_{sjt}. \tag{19}$$

Since market shares sum to one, the fringe firms' share is $1 - \phi_{sjt}$. The equilibrium price of the superstar firm under Bertrand competition is[10]

$$p_{sjt} = \frac{1 - \varepsilon\, \phi_{sjt}}{(1 - \phi_{sjt})\, \varepsilon} MC_{sjt}, \tag{20}$$

where MC_{sjt} denotes the marginal cost of the superstar firm.[11] The equilibrium price is increasing in product quality and the embedded intangible stock, and decreasing in the number of sectors operated. The price ratio of fringe to superstar firms is

$$\frac{p_{fjt}}{p_{sjt}} = \frac{(1 - \phi_{sjt})\varepsilon}{1 - \varepsilon\phi_{ijt}} \cdot \lambda^{m_j} \left(\frac{(1-\xi)e_{st}}{n_s} \right)^{\alpha}. \tag{21}$$

Substituting the ideal price index from equation (18) into the market share definition in equation (19), the superstar's market share can be expressed in terms of relative prices as

$$\phi_{sjt} = \frac{1}{1 + \left(\frac{1}{(\xi e_{st})^{\frac{\beta}{1-\varepsilon}}} \left(\frac{p_{fjt}}{p_{ijt}} \right)^{\frac{\varepsilon}{\varepsilon-1}} \right)}. \tag{22}$$

Equation (22) shows that the superstar firm's market share depends on the quality gap m_j, the embedded intangible stock e_{st}, and the number of sectors operated n_s.

[10] See Appendix B.3 for the full derivations. For the Cournot competition version, see Appendix B.4.

[11] Although fringe firms lack independent pricing power, their presence creates a competitive constraint that disciplines the superstar firm, preventing it from fully extracting monopolistic rents.

Profit, Markup, and Labor Demand. The static operational profit and markup of the superstar firm are proportional to its market share:

$$\pi_{sjt} = \frac{(1-\varepsilon)\,\phi_{sjt}}{1-\varepsilon\phi_{sjt}}\,Y_t \qquad \text{and} \qquad \sigma_{sjt} = \frac{1-\varepsilon\phi_{sjt}}{(1-\phi_{sjt})\,\varepsilon}. \tag{23}$$

Both markup and profit increase with market share and therefore decline with firm scope, reflecting the diminishing managerial productivity that arises from the span-of-control friction. The optimal labor inputs for superstar and fringe firms are

$$l_{sjt} = \frac{\phi_{sjt}}{\sigma_{sjt}}\,\omega_t^{-1} \quad \text{and} \quad l_{fjt} = (1-\phi_{sjt})\,\omega_t^{-1}, \tag{24}$$

where $\omega_t = w_t/Y_t$ denotes the wage share of the economy.

The static equilibrium provides only an implicit solution. Nonetheless, the model yields tractable dynamics because the equilibrium outcome for a superstar firm depends solely on its market share, which is determined by the quality gap, the embedded intangible stock, and the number of sectors it operates in.

Superstar Value Function. The value function of superstar firm s depends on the quality gap vector $\mathbf{m} = \{m_j\}_{j=1}^{n}$, the embedded intangible gap k, and the number of sectors operated n.[12] The firm chooses innovation flow rates z_j^{Ver}, z_j^{Hor}, and z_j^{Emb} to maximize

$$\begin{aligned}
r_t V_t(\mathbf{m},k,n) - \dot{V}_t(\mathbf{m},k,n) = \max_{z_{jt}^{\text{Ver}},z_{jt}^{\text{Hor}},z_{jt}^{\text{Emb}}} \sum_{j=1}^{n} \Bigg(&\frac{(1-\varepsilon)\,\phi_t(m_j,k,n)}{1-\varepsilon\,\phi_t(m_j,k,n)}\,Y_t \\
&+ \underbrace{\mathbb{P}_{s\geq s'}\, z_{jt}^{\text{Hor}}\Big[V_{jt}\big((m_j,1),k,n+1\big) - V_{jt}(m_j,k,n)\Big]}_{\text{Horizontal Innovation}} + \underbrace{z_{jt}^{\text{Ver}}\Big[V_{jt}(m_j+1,k,n) - V_{jt}(m_j,k,n)\Big]}_{\text{Vertical Innovation}} \\
&+ \underbrace{z_{jt}^{\text{Emb}}\Big[V_{jt}(\mathbf{m},k+1,n) - V_{jt}(\mathbf{m},k,n)\Big]}_{\text{Embedded Intangible Innovation}} + \underbrace{Z_{(c>s)\,t}^{\text{Hor}}\Big[-V_{jt}(m_j,k,n)\Big]}_{\text{Challenger Entry}} + \underbrace{\mathbb{I}_{f>s}\,Z_{jt}^{f}\Big[-V_{jt}(m_j,k,n)\Big]}_{\text{Fringe Entry}} \\
&- \gamma^{\text{Ver}}\left(z_{jt}^{\text{Ver}}\right)^{\vartheta^{\text{Ver}}} Y_t - \gamma^{\text{Emb}}\left(z_{jt}^{\text{Emb}}\right)^{\vartheta^{\text{Emb}}} Y_t - \gamma^{\text{Hor}}\left(z_{jt}^{\text{Hor}}\right)^{\vartheta^{\text{Hor}}} Y_t \Bigg).
\end{aligned} \tag{25}$$

[12]To simplify notation, the subscript s is omitted whenever it is clear from context.

The left-hand side of the value function shows the return on the value function and its gain over time. The first line on the right-hand side represents the profit of the superstar firm in sector j. The second and third terms capture the value gains from horizontal and vertical innovation, respectively: a successful horizontal innovation expands the firm's scope from n to $n+1$, while a successful vertical innovation advances the quality gap in sector j by one rung. The fourth term captures the firm-wide gain from a successful embedded intangible investment, which raises the embedded gap k by one rung across all active sectors simultaneously. The fifth and sixth terms capture the value loss upon displacement by a challenger superstar or a fringe firm. The final three terms are the convex investment costs defined in equation (8). The terms $Z^{\text{Hor}}_{(c>s)\,t}$ and Z^{f}_{jt} denote the aggregate innovation rates of challenger superstar firms and fringe firms, respectively. The aggregate horizontal innovation rate of challenger superstars that successfully displace the incumbent is

$$Z^{\text{Hor}}_{(c>s)\,t} = \sum_{m_c,\,k_c,\,n_c} \mathbb{I}\big\{\lambda\,(\theta^{k_c})^{\alpha}\,n_{ct}^{1-\alpha} > (\theta^{k_s})^{\alpha}\,n_{st}^{1-\alpha}\big\}\,z^{\text{Hor}}_t(m_c,k_c,n_c)\,\mu_t(m_c,k_c,n_c),$$

and the aggregate fringe firm innovation rate is $Z^{f}_{jt} = \int_0^1 z_{ijt}\,di$.

Along the balanced growth path, aggregate output Y, consumption C, and the value function $V(\mathbf{m},k,n)$ all grow at the constant rate g. Defining the normalized value function $v(\mathbf{m},k,n) \equiv V(\mathbf{m},k,n)/Y$, the stationary value function satisfies

$$\begin{aligned}\rho\,v(\mathbf{m},k,n) = \max_{z_j^{\text{Ver}},z_j^{\text{Hor}},z_j^{\text{Emb}}} \sum_{j=1}^{n} \Bigg(& \frac{(1-\varepsilon)\,\phi(m_j,k,n)}{1-\varepsilon\,\phi(m_j,k,n)} \\ & + \underbrace{\mathbb{P}_{s\geq s'}\,z_j^{\text{Hor}}\Big[v\big((m_j,1),k,n+1\big)-v(m_j,k,n)\Big]}_{\text{Horizontal Innovation}} + \underbrace{z_j^{\text{Ver}}\Big[v(m_j+1,k,n)-v(m_j,k,n)\Big]}_{\text{Vertical Innovation}} \\ & + \underbrace{z_j^{\text{Emb}}\Big[v(\mathbf{m},k+1,n)-v(\mathbf{m},k,n)\Big]}_{\text{Embedded Intangible Innovation}} + \underbrace{Z^{\text{Hor}}_{c>s}\Big[-v(m_j,k,n)\Big]}_{\text{Challenger Entry}} + \underbrace{\mathbb{I}_{f>s}\,Z^{f}_{j}\Big[-v(m_j,k,n)\Big]}_{\text{Fringe Entry}} \\ & - \gamma^{\text{Ver}}\left(z_j^{\text{Ver}}\right)^{\vartheta^{\text{Ver}}} - \gamma^{\text{Emb}}\left(z_j^{\text{Emb}}\right)^{\vartheta^{\text{Emb}}} - \gamma^{\text{Hor}}\left(z_j^{\text{Hor}}\right)^{\vartheta^{\text{Hor}}}\Bigg). \end{aligned} \tag{26}$$

Fringe Value Function. Since fringe firms produce a homogeneous good, they earn zero profit in equilibrium and derive value solely from the prospect of displacing the incumbent through radical innovation. The fringe firm chooses innovation intensity z_{ft} to maximize this prospect, trading off investment costs against the value gain from becoming the new superstar. The indicator $\mathbb{I}_{f>s}$ equals one when the fringe firm's drastic innovation is sufficient to overcome the incumbent's total managerial capacity and displace it. The value function also accounts for changes in the fringe firm's competitive environment arising from the incumbent superstar's own vertical, horizontal, and embedded intangible innovations, from the entry of a challenger superstar that displaces the incumbent, and from the innovation of another fringe firm that successfully becomes the new superstar, each of which alters the state the fringe firm faces.

$$\begin{aligned} r_t V_{ft}(m,k,n) - \dot{V}_{ft}(m,k,n) = \max_{z_{ft}} \Bigg\{ & \underbrace{\mathbb{I}_{f>s'}\, z_{ft}\Big[V_t(\bar{m},1,1) - V_{ft}(m,k,n)\Big]}_{\text{Radical Innovation}} - \gamma^f (z_{ft})^{\vartheta f} Y_t \\ & + \underbrace{\mathbb{P}_{s\geq s'}\, z_{jt}^{\text{Hor}}\Big[V_{ft}\big((m,1),k,n+1\big) - V_{ft}(m,k,n)\Big]}_{\text{Incumbent Horizontal Innovation}} + \underbrace{z_{jt}^{\text{Ver}}\Big[V_{ft}(m+1,k,n) - V_{ft}(m,k,n)\Big]}_{\text{Incumbent Vertical Innovation}} \\ & + \underbrace{z_{jt}^{\text{Emb}}\Big[V_{ft}(m,k+1,n) - V_{ft}(m,k,n)\Big]}_{\text{Incumbent Embedded Innovation}} + \underbrace{\mathbb{E}_{\mu_t}\big[\mathbb{I}_{c>s}\cdot z_t^{\text{Hor}}\cdot \Delta V_{ft}\big]}_{\text{Challenger Superstar Entry}} + \\ & \underbrace{\mathbb{I}_{f>s}\, Z_{jt}^{f}\Big[V_{ft}(\bar{m},1,1) - V_{ft}(m,k,n)\Big]}_{\text{Fringe Challenger}} \Bigg\}, \end{aligned} \tag{27}$$

where the challenger superstar entry term is

$$\mathbb{E}_{\mu_t}\big[\mathbb{I}_{c>s}\cdot z_t^{\text{Hor}}\cdot \Delta V_{ft}\big] \equiv \sum_{m_c,k_c,n_c} \mathbb{I}\big\{\lambda(\theta^{k_c})^{\alpha} n_c^{1-\alpha} > (\theta^{k_s})^{\alpha} n_s^{1-\alpha}\big\}\; z_t^{\text{Hor}}(m_c,k_c,n_c)\; \mu_t(m_c,k_c,n_c)\Delta V_{ft},$$

with $\Delta V_{ft} \equiv V_{ft}(1,k_s,n_s) - V_{ft}(m,k,n)$ denoting the value change for the fringe firm upon facing a new incumbent drawn from μ_t.

Innovation Decisions. The first-order conditions of the stationary value functions yield the following optimal innovation rates:

$$z_j^{\text{Ver}} = \left(\frac{v_j(m_j+1,k,n) - v_j(m_j,k,n)}{\gamma^{\text{Ver}} \cdot \vartheta^{\text{Ver}}} \right)^{\frac{1}{\vartheta^{\text{Ver}}-1}}, \tag{28}$$

$$z_j^{\text{Hor}} = \left(\frac{\mathbb{P}_{s\geq s'} \left[v_j\big((m_j,1),k,n+1\big) - v_j(m_j,k,n) \right]}{\gamma^{\text{Hor}} \cdot \vartheta^{\text{Hor}}} \right)^{\frac{1}{\vartheta^{\text{Hor}}-1}}, \tag{29}$$

$$z_j^{\text{Emb}} = \left(\frac{v(\mathbf{m},k+1,n) - v(\mathbf{m},k,n)}{\gamma^{\text{Emb}} \cdot \vartheta^{\text{Emb}}} \right)^{\frac{1}{\vartheta^{\text{Emb}}-1}}, \tag{30}$$

$$z_f = \left(\frac{\mathbb{I}_{f>s} \left[v(\bar{m},1,1) - v_f(m,k,n) \right]}{\gamma^{f} \cdot \vartheta^{f}} \right)^{\frac{1}{\vartheta^{f}-1}}. \tag{31}$$

Each optimal innovation rate is increasing in the value gain from the corresponding innovation and decreasing in its cost scale and curvature parameters.

Distribution Evolution. The law of motion of the distribution $\mu_t(m,k,n)$ is driven by net flows into and out of state (m,k,n). Inflows arise from firms that enter (m,k,n) following a successful innovation in one or more dimensions; outflows arise from firms that leave (m,k,n) as a result of their own innovation or displacement by entrants.

$$\begin{aligned}
\dot{\mu}_t(m,k,n) = {} & z_t^{\text{Ver}}(m-1,k,n)\,\mu_t(m-1,k,n) + z_t^{\text{Emb}}(m,k-1,n)\,\mu_t(m,k-1,n) \\
& + \mathbb{P}_{s\geq s'}\, z_t^{\text{Hor}}(m,k,n-1)\,\mu_t(m,k,n-1) \\
& + z_t^{\text{Ver}}(m-1,k-1,n)\, z_t^{\text{Emb}}(m-1,k-1,n)\,\mu_t(m-1,k-1,n) \\
& + z_t^{\text{Ver}}(m-1,k,n-1)\,\mathbb{P}_{s\geq s'}\, z_t^{\text{Hor}}(m-1,k,n-1)\,\mu_t(m-1,k,n-1) \\
& + z_t^{\text{Emb}}(m,k-1,n-1)\,\mathbb{P}_{s\geq s'}\, z_t^{\text{Hor}}(m,k-1,n-1)\,\mu_t(m,k-1,n-1) \\
& + z_t^{\text{Ver}}(m-1,k-1,n-1)\, z_t^{\text{Emb}}(m-1,k-1,n-1) \\
& \times\ \mathbb{P}_{s\geq s'}\, z_t^{\text{Hor}}(m-1,k-1,n-1)\,\mu_t(m-1,k-1,n-1) \\
& - z_t^{\text{Ver}}(m,k,n)\,\mu_t(m,k,n) - z_t^{\text{Emb}}(m,k,n)\,\mu_t(m,k,n) \\
& - \mathbb{P}_{s\geq s'}\, z_t^{\text{Hor}}(m,k,n)\,\mu_t(m,k,n) - \mathbb{I}_{f>s}\, Z_t^{f}(m,k,n)\,\mu_t(m,k,n) - Z_{(c>s\,t)}^{\text{Hor}}\,\mu_t(m,k,n)
\end{aligned} \tag{32}$$

The first three terms capture inflows from states one step behind in a single dimension; the next three capture inflows from states one step behind in two dimensions; and the subsequent term captures inflows from states one step behind in all three dimensions simultaneously. The final group of terms captures outflows due to the incumbent's own innovation, displacement by a fringe firm, and displacement by a challenger superstar.[13]

Aggregate Variables. The labor market clears according to

$$1 = \int_0^1 \left(l_{sjt} + l_{fjt}\right) dj. \tag{33}$$

Using equations (24) and (33), the normalized wage satisfies

$$\omega_t = \sum_{m=1}^{\bar{m}} \sum_{k=1}^{\bar{k}} \sum_{n=1}^{\bar{n}} \left(\tfrac{\phi_t(m,k,n)}{\sigma_t(m,k,n)} + 1 - \phi_t(m,k,n) \right) \mu_t(m,k,n). \tag{34}$$

Combining the production functions (3), (6), and (7) with the labor demand equations (24) yields aggregate output

$$Y_t = Q_t \, \omega_t^{-1} \, \exp\left(\sum_{m=1}^{\bar{m}} \sum_{k=1}^{\bar{k}} \sum_{n=1}^{\bar{n}} \underbrace{\ln \left[\left(\xi \theta^{k_t}\right)^{\beta} \left(\frac{((1-\xi)\theta^{k_t})^{\alpha}}{\gamma n_t^{\alpha_s}} \frac{\phi_t(m,k,n)}{\sigma_t(m,k,n)} \right)^{\varepsilon} + \left(\lambda^{-m_t} (1 - \phi_t(m,k,n)) \right)^{\varepsilon} \right]^{\frac{1}{\varepsilon}}}_{\equiv R_t(m,k,n)} \mu_t(m,k,n) \right), \tag{35}$$

where $Q_t = \exp\Big(\int_0^1 \ln q_{sjt} \, dj \Big)$. Along the balanced growth path, the economy grows at rate[14]

$$g \;=\; \ln \lambda \sum_{m=1}^{\bar{m}} \sum_{k=1}^{\bar{k}} \sum_{n=1}^{\bar{n}} \left(z^{\text{Ver}}(m,k,n) + \mathbb{P}_{s \geq s'} \, z^{\text{Hor}}(m,k,n) + Z^f(m,k,n) \right) \mu(m,k,n). \tag{36}$$

The resource constraint is

$$Y_t = C_t + \int_0^1 \left(I_{jt}^{\text{Ver}} + I_{jt}^{\text{Hor}} + I_{jt}^{\text{Emb}}\right) dj + \int_0^1 I_{jt}^f \, dj, \quad \text{with} \quad I_{jt}^f = \int_{F_j} I_{ijt} \, di. \tag{37}$$

Equilibrium Definition. A Markov Perfect equilibrium consists of an allocation $\{C_t, Y_t, y_{sjt}, y_{fjt}\}$, prices $\{r_t, w_t, p_{sjt}, p_{fjt}\}$, and innovation policies $\{z^{\text{Ver}}, z^{\text{Hor}}, z^{\text{Emb}}, z_f\}$ and labor allocations $\{l_{sjt}, l_{fjt}\}$ and distribution $\mu_t(m,k,n)$ such that: the final good producer maximizes profit

[13] At the boundaries of the state space, the relevant outflow terms are set to zero. When any dimension attains its upper bound $(\bar{m}, \bar{k}, \bar{n})$, outflows from further innovation in that dimension are zero. When any dimension attains its lower bound $(1,1,1)$, inflows from lower states are zero.

[14] See Appendix B.5 for details.

given prices; the superstar firm maximizes its value function given its state $(\mathbf{m}, k, n)$, choosing vertical, horizontal, and embedded innovation rates as in equations (28)–(30); the fringe firm chooses its optimal innovation rate as in equation (31); the labor market clears; $\mu_t(m, k, n)$ satisfies the each period equation (14) and evolves according to equation (32); aggregate consumption and output grow at the common rate g given by equation (36), with the resource constraint (37) satisfied.

3. Dataset and Empirical Facts

This section describes the data sources and variable construction and documents a set of stylized facts on how firm scope shapes the transferable-to-embedded intangible investment ratio, competitive threat, productivity, and markups.

3.1. Data Description

Compustat Fundementals and Segment. Compustat Fundamentals provides comprehensive firm-level financial information for publicly listed companies in North America and offers extensive longitudinal coverage. It includes detailed balance sheet items, income statement components, cash flow data, and key financial ratios. An additional advantage is that it enables data to be merged with external datasets through a unique firm identifier.

Measuring firm scope requires information on the industries in which a firm operates. Compustat provides two segment datasets: (i) the Historical Segment dataset[15], which offers long-run coverage of firms' segment activities but limited segment-level detail, and (ii) the Compustat Segment dataset, introduced in 2016, which provides richer information on segment characteristics but over a shorter time horizon. In both datasets, a firm's segments are defined as the 2-digit industries in which it operates.[16]

[15] Under Regulation SFAS No. 131—codified as ASC 280 after 2009—U.S. public firms are required to disclose the identity of any one customer that accounts for more than 10% of its total revenue, along with the nature of the products or services provided to that customer. These mandated disclosures constitute the foundation of the Historical Segment datasets.

[16] For a comparative illustration of firm segment classification, see Tables A1.

To balance coverage and detail, I combine the two sources, relying on the Compustat Segment dataset when available and the Historical Segment dataset otherwise.[17] From these data, I identify the set of two-digit industries in which each firm operates in a given year and define firm scope as the number of such industries. I then merge the segment information with Compustat Fundamentals to construct firm-level measures of markups, productivity, and the composition of intangible investment across sectors.[18]

Fludity Dataset. The product market fluidity metric from Hoberg, Phillips, and Prabhala (2014) measures the rate at which firms in similar markets change their product or service offerings annually. It is calculated using natural language processing (NLP) on the product descriptions from firms' annual 10-K reports filed with the U.S. Securities and Exchange Commission. This method tracks year-over-year changes in how companies describe their business. A high fluidity score indicates that competitors are adapting rapidly by launching new products, shifting strategies, and entering new markets. Consequently, firms in high-fluidity markets face heightened competitive threats from rivals reconfiguring their offerings and positions.

3.2. Measurement

Productivity and Markup Estimation. I estimate firm-level total factor productivity using the approach developed by Gandhi, Navarro, and Rivers (2020)[19]. They propose a nonparametric identification strategy that uses a transformation of the first-order condition for intermediate inputs to isolate flexible-input effects and identify the production func-

[17]In this project, for the Compustat Historical Segment dataset, geographic and operational segments are excluded; only business segments are retained. For the Compustat Segment dataset, only non-missing entries from the Product–Service (PD–SRVC) category are included.When segment information for a firm is available in the Compustat Segment dataset, I prioritize that source. Otherwise, I use data from the Historical Segment dataset.

[18]The merged Compustat dataset contains fewer firms than the Fundamentals database because segment information is unavailable for some firms. In addition, I restrict the sample to firms with positive R&D and SG&A expenditures. This cleaning and merging process does not affect the representativeness of the merged dataset relative to the full Compustat Fundamentals sample; see Figure A3.

[19]The productivity estimation results are robust to alternative production function estimation methods, including those proposed by Ackerberg, Caves, and Frazer (2015) and Levinsohn and Petrin (2003); see Figure A2. For methodological details, see Appendix A.1

tion and input elasticities without relying solely on proxy inversion. To estimate firm-level markups, I follow the methodology of De Loecker, Eeckhout, and Unger (2020) and define markups as the ratio of sales to the cost of goods sold (cogs) multiplied by the output elasticity of the variable input, which I obtain from the first-stage production function estimation using Levinsohn and Petrin, 2003.

Investment Ratio. Following Peters and Taylor (2017), I measure transferable intangible investment as total R&D expenditure and embedded intangible investment as 30% of Selling, General, and Administrative (SG&A) expenses net of R&D. SG&A encompasses a broad range of expenditures including advertising, employee compensation, and general operational costs. Once R&D is netted out, the remaining SG&A primarily reflects advertising spending and employee-related costs, the majority of which are routine operating expenses consumed in the current period. However, a portion of these expenditures accumulates within the firm as lasting intangible capital: advertising builds brand recognition and employee compensation develops organizational knowledge and managerial talent. The 30% fraction captures this accumulating component, with the remainder treated as routine operating costs that generate no lasting intangible value.

3.3. Empirical Facts

To document how firm-level outcomes respond to innovation, I estimate local projections following Jordà (2005) with the patent-value measure of Kogan, Papanikolaou, Seru, and Stoffman (2017) as an innovation shock. Patent value captures the excess stock market reaction to patent grants within a narrow event window around the grant date, reflecting the market's assessment of the unexpected returns to the innovation. A key feature of this measure is that it arrives at the grant date, which is largely predetermined relative to the firm's strategic decisions.[20] The local projection specification takes the form

$$\Delta_h Y_{ijt} = \alpha_h + \beta_{\text{Multi}} V_{it} D_{it} + \beta_{\text{Single}} V_{it}(1 - D_{it}) + \boldsymbol{\Gamma}_h^\top \mathbf{X}_{i,t-1} + \delta_h Y_{i,t-1} + \theta_j + \lambda_t + u_{ijt}, \quad (38)$$

where $\Delta_h Y_{ijt} \equiv Y_{i,t+h} - Y_{it}$ is the h-period change in outcome Y for firm i in industry j, $V_{it} = \sum_{p \in \mathcal{P}_{it}} \text{PatentValue}_{p,t}$ is the firm-year sum of patent value shocks, and $D_{it} = \mathbf{1}\{i \text{ is multi-sector at } t\}$ indicates multi-sector status. The vector $\mathbf{X}_{i,t-1}$ collects lagged controls, $Y_{i,t-1}$ absorbs pre-shock outcome levels, and θ_j and λ_t denote industry and year fixed effects. The coefficients β_{Single} and β_{Multi} trace the impulse responses of single- and multi-sector firms, respectively, to an equivalent innovation shock. Scope status is defined using each firm's pre-shock classification and held fixed throughout the estimation window, ensuring that firms that subsequently change their reported segments do not mechanically alter the interaction term.

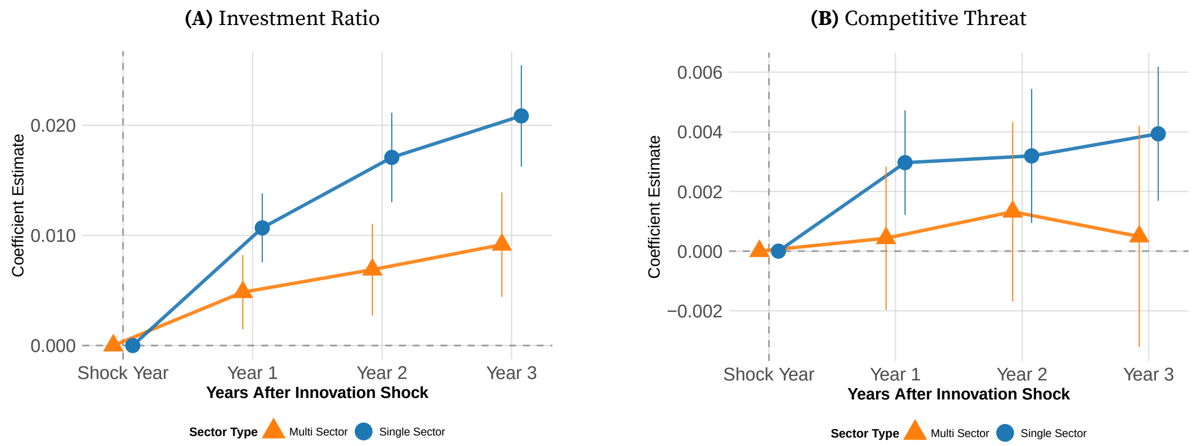

Figure 2. Markup, Investment Ratio, and Fluidity with Production Lines

Note: The sample excludes utilities and finance sectors, as well as firms with missing or non-positive R&D and SG&A. All variables are in logarithmic form and cover the period 1990–2019.

Figure 2(A) establishes that innovation shocks increase the investment ratio for both single- and multi-sector firms, reflecting a reallocation of resources toward knowledge-generating intangibles, but the magnitude and precision of the response differ markedly

[20]Segment changes are relatively infrequent in the data, suggesting that the innovation shocks identified here primarily reflect vertical improvements within existing sectors rather than scope-expanding horizontal innovations.

across firm types. Single-sector firms exhibit a stronger and more precisely estimated investment response, with the gap opening at the shock year and widening monotonically over the subsequent three years. This divergence suggests that operating across multiple sectors weakens the incentive to shift investment toward transferable intangibles: when innovative activity is concentrated in a single domain, the patent-value shock translates into a larger and more sustained reallocation away from embedded toward transferable intangible investment.

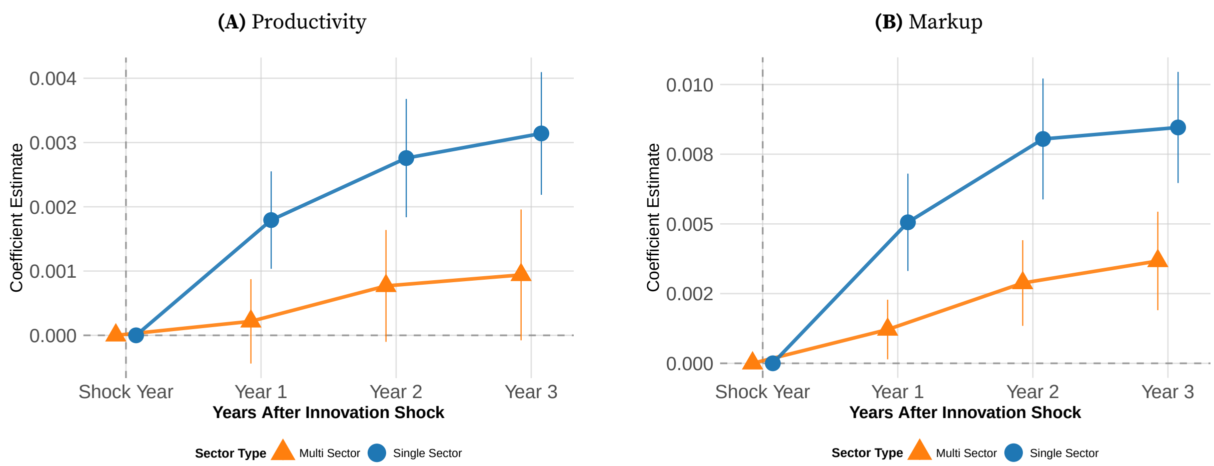

Figure 3. Markup, Investment Ratio, and Fluidity with Production Lines

Note: The sample excludes utilities and finance sectors, as well as firms with missing or non-positive R&D and SG&A. All variables are in logarithmic form and cover the period 1990–2019.

Figure 2(B) shows that single-sector innovators attract substantially greater competitive entry following an innovation shock. This result indicates that innovation does not remain internal to the firm; instead, it diffuses outward, reshaping the competitive landscape and inducing rival entry. By contrast, the response for multi-sector firms remains close to zero throughout the horizon and is statistically indistinguishable from zero at all horizons. This near-zero response in competitive threat is itself informative. One interpretation is that multi-sector firms create stronger entry barriers in the markets in which they operate, limiting the diffusion of knowledge to rivals. Alternatively, innovation in multi-sector firms may rely on cross-sector complementarities that rivals with a narrower scope cannot easily replicate. Either mechanism implies that multi-sector firms appropriate a larger share of the gains from their innovations internally.

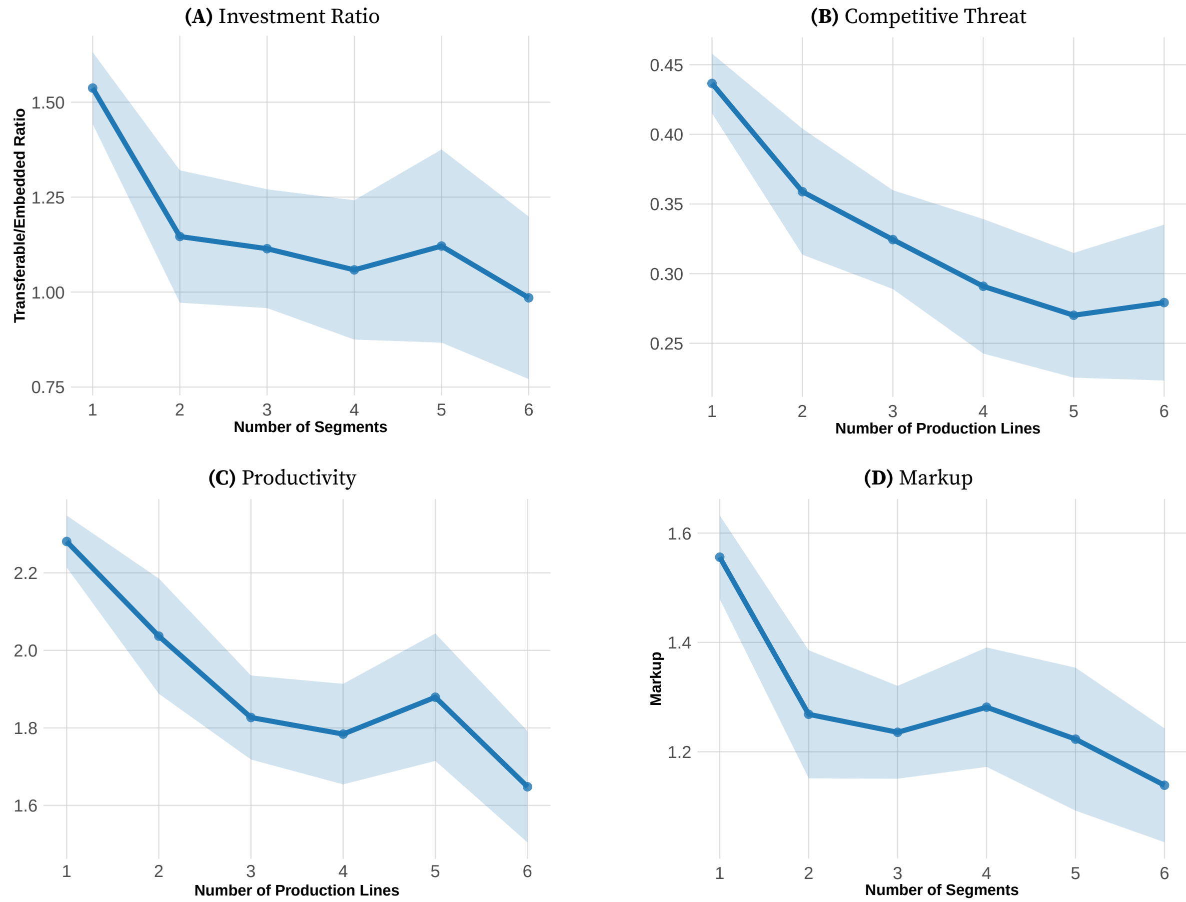

Figure 4. Innovation Shock to Single and Multi-Segment Firms

Note: The sample excludes utilities and finance sectors, and firms with missing or non-positive R&D and SG&A. Markups, the investment ratio, and market fluidity are measured for the 2019 cross-section. The investment ratio is transferable over embedded investment, which is described in Section 3.2. The investment ratio is winsorized at the 95th percentile, the markup at the 90th percentile, and labor market fluidity at the top and bottom 5th percentiles. For calibration purposes, fluidity is normalized using min-max scaling. Each value x was transformed according to $x_{\text{scaled}} = (x - \min(x))/(\max(x) - \min(x))$, mapping all values linearly into the range $[0, 1]$.

Figure 3 (A) and Figure 3 (B) show that the heterogeneity between firm types persist also productivity and markups that more directly capture the appropriability of innovation rents. Single-sector firms display the highest productivity levels and the strongest productivity responses following an innovation shock, while multi-sector firms exhibit considerably weaker responses. A similar pattern emerges for markups. Firms with narrower scope command higher price–cost margins, and their markups rise more strongly following innovation shocks. This finding is consistent with Autor, Dorn, Katz, Patterson, and Van Reenen (2020) in which higher-productivity firms charge higher markups. In contrast, multi-sector firms display muted responses in both productivity and markups, and the estimates are less precisely measured. Taken together, Figure 2 and Figure 3 show that firm scope systematically moderates the returns to innovation along every margin. Because the Kogan, Papanikolaou, Seru, and Stoffman (2017) patent-value measure is constructed to be directly comparable across firm types, the differential responses are unlikely to reflect differences in shock quality or industry composition.

Figure 4 complements local projection patterns in the cross-section. Across all four panels, the transferable-to-embedded investment ratio, competitive threat, productivity, and markups decline monotonically as the number of sectors increases. Table A3 in appendix confirms that these relationships are not driven by industry composition or aggregate time trends: two-way fixed effects estimates remain negative and statistically significant across all outcomes, indicating that the scope relationship reflects systematic within-firm variation. In particular, the cross-sectional moments documented in Figure 4 serve as calibration targets in the simulated method of moments estimation of Section 4, where I discipline the steady-state properties of the model. The local projections, cross-sectional patterns, and panel regressions point to a coherent set of regularities, summarized by the following four stylized facts.

Fact 1. The transferable-to-embedded intangible investment ratio declines monotonically with firm scope, both in the cross-section and within firms over time. Following an innovation shock, single-sector firms reallocate toward transferable intangibles more strongly and persistently than multi-sector firms.

Fact 2. Competitive threat declines with firm scope. Single-sector firms attract measurable rival entry following an innovation shock, while multi-sector firms exhibit no significant competitive response. This pattern holds within firms over time after controlling for industry and year fixed effects.

Fact 3. Productivity declines monotonically with the number of production lines. Single-sector firms exhibit the strongest productivity gains following an innovation shock, while the multi-sector response is weak.

Fact 4. Markups decline monotonically with firm scope. Single-sector firms charge the highest markups and exhibit the largest markup responses following an innovation shock, while multi-sector firms display modest responses, indicating that scope limits the conversion of innovation into pricing power.

4. Quantitative Analysis

This section presents the calibration of the model and examines its quantitative implications. I calibrate the model using the simulated method of moments, targeting firm-level moments on markups, the transferable-to-embedded intangible investment ratio, and the distribution of firm scope. The model's fit is then assessed against untargeted moments on productivity growth, competitive threat, and the growth rate. Section 4.2 analyzes how the composition of intangible investment shapes innovation direction and firm dynamics. Section 4.3 examines the role of the relative shares of brand value and organizational capital in determining firms' incentives to expand scope.

4.1. Calibration and Model Performance

The model is disciplined by 36 empirical moments, comprising 18 targeted and 18 untargeted moments. The calibration relies on 16 parameters, of which 4 are set externally. The time discount rate is fixed at $\rho = 0.05$, and the curvature parameters governing R&D investment for vertical innovation, horizontal innovation, and fringe firms are set to $\vartheta^{\text{Ver}} = \vartheta^{\text{Hor}} = \vartheta^{f} = 2.0$, following Akcigit and Kerr (2018). The remaining 12 parameters,

$$\{\varepsilon, \lambda, \theta, \alpha, \gamma, \beta, \xi, \gamma^{\text{Ver}}, \gamma^{\text{Hor}}, \gamma^{f}, \gamma^{\text{Emb}}, \vartheta^{\text{Emb}}\},$$

are estimated internally via the simulated method of moments. These parameters govern the key structural features of the model, and all estimated values are reported in Table 1.[21]

[21] Appendix C.2 details the construction of the model's empirical counterparts, including the aggregation of the joint distribution $\mu(m, k, n)$ to scope level n, and the derivation of competitive threat, markups, the investment ratio, productivity, and growth rates as functions of firm scope.Appendix C.3 provides details

Table 1. Parameter Values

Parameter	Description	Value
	———————*External Calibration*———————	
ρ	Discount rate	0.05
ϑ^{Ver}	Curvature of vertical investment	2.0
ϑ^{Hor}	Curvature of horizontal investment	2.0
ϑ^{f}	Curvature of fringe firms investment	2.0
	———————*Internal Calibration*———————	
ε	CES parameter	0.7747
λ	Quality improvement step size	1.0100
θ	Embedded innovation step size	1.1625
α	Curvature of managerial quality	0.5980
γ	Scale of managerial quality	0.4001
β	Curvature of brand value	0.0661
ξ	Share of brand value	0.4351
γ^{Ver}	Cost scale of internal innovation	2.8571
γ^{Hor}	Cost scale of horizontal innovation	0.4001
γ^{f}	Cost scale of fringe	5.1777
γ^{Emb}	Cost scale of embedded innovation	0.0664
ϑ^{Emb}	Curvature of embedded innovation	3.3646

Note: The upper limit for the number of production lines $\bar{n}$ is set to 6, and the upper bounds for $\bar{m}$ and $\bar{k}$ are set to 9.

Figure 5 reports the targeted moments, focusing on markup dynamics, the ratio of transferable-to-embedded intangible investment, and the distribution of firms across production lines. Overall, the model reproduces the principal empirical patterns: it captures both the direction and magnitude of the observed trends. Panel (A) shows that the model tracks the decline in markups as the number of production lines increases, although modest deviations remain at the extremes of the distribution—the model understates markups for single-line firms and slightly overstates them for firms operating many lines. Panel (B) illustrates that the transferable-to-embedded investment ratio is well matched across production-line categories, with simulated moments closely following the empirical shape. Panel (C) demonstrates a strong fit for the firm distribution,

of the solution algorithm and the simulated method of moments.

in which simulated shares align closely with observed data. Taken together, these results suggest that the model is well disciplined by the targeted moments and captures key margins of firm behavior, with only minor discrepancies concentrated in the tails of the markup profile

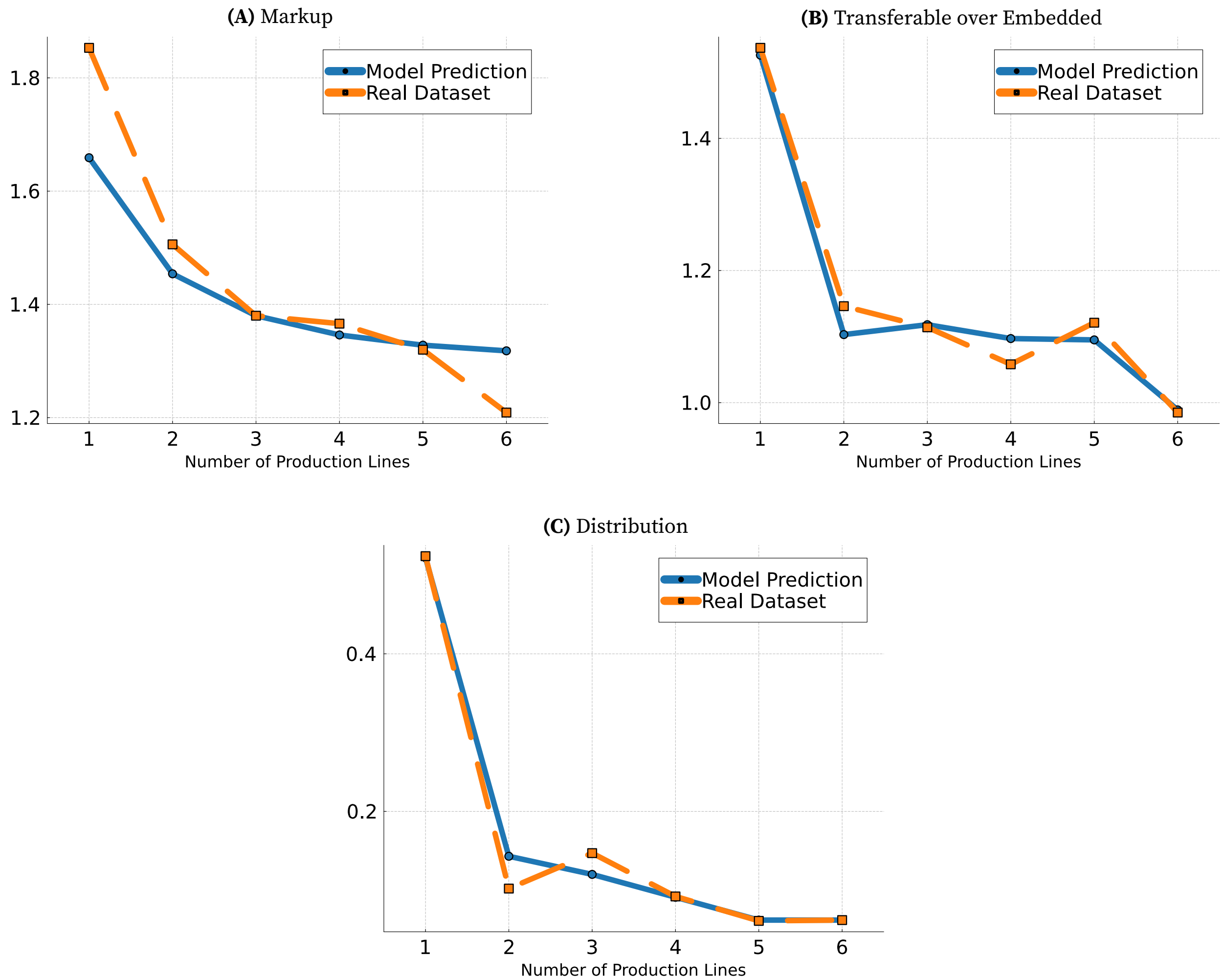

Figure 5. Targeted Moments: Markup, Investment Ratio, and Fluidity by Production Lines

Note: The orange line represents the dataset values, while the blue line shows the model simulation results along the balanced growth path. The horizontal axis corresponds to the production line dimension.

Further, Figure C1 evaluates the model's performance on untargeted moments, including productivity, aggregate growth rates, and measures of fluidity across production-line categories. Panel (A) indicates that the model captures the declining pattern of productivity as production-line count rises, with only small departures from the data at intermediate values. Panel (B) reveals systematic differences in the aggregate growth rate: the model tends to underestimate actual growth for firms with one to three production lines and slightly overshoots growth at the upper end of the distribution. These discrepancies imply that, while the model reproduces the overall downward growth trend, it misses some non-monotonic features present in the data. Panel (C) shows that the model generally undershoots empirical fluidity measures across most production-line categories, with the notable exception of the sixth line, where simulated fluidity converges more closely to the observed value.

4.2. Firm Innovation Direction Decision

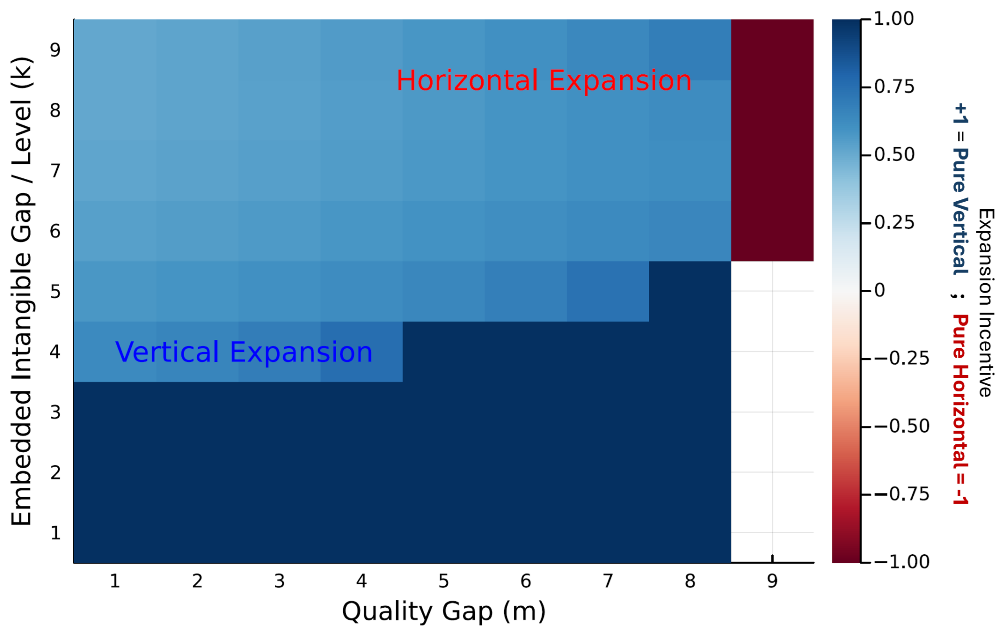

Figure 6. Superstar Firms Innovation Direction (Vertical vs Horizontal)

Figure 6 illustrates a firm's optimal innovation direction as a function of the quality gap m and the embedded-intangible gap k. The horizontal axis reports the firm's position on the quality gap relative to competitors (proportional to the superstar's transferable intangible level), while the vertical axis reports embedded intangible levels. The color scale shows the normalized innovation incentive: darker blue denotes a stronger incentive for vertical innovation, darker red denotes a stronger incentive for horizontal innovation, and intermediate shades indicate regions where the two strategies yield comparable payoffs.

The figure exhibits a clear pattern. For low values of k, the firm optimally focuses on vertical innovation: embedded assets are insufficient to generate substantial cross-sector synergies, so the marginal return to improving existing products exceeds the return to expanding scope. As k increases, embedded intangibles amplify cross-sector complementarities and raise the payoff to horizontal expansion. Beyond a threshold level of k, horizontal innovation becomes the preferred strategy even when the quality gap m is moderate. Moreover, for a given k, a larger quality gap m shifts the boundary toward vertical innovation. The implication is that firms of similar size may follow different innovation strategies depending on the composition of their intangible endowments: size alone does not determine innovation direction—whether a firm invests more in transferable or embedded intangibles matters critically.

4.3. Demand and Supply Effect of Embedded Intangibles

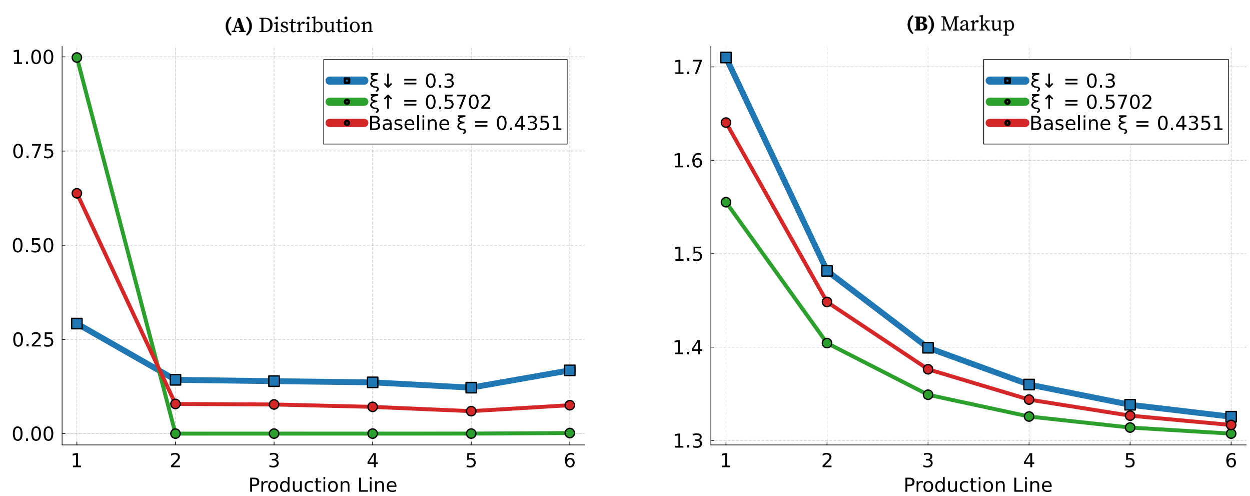

Figure 7. Impact of ξ on Distribution and Markup

Note: The green line represents the internally calibrated optimal value of ξ, while the blue line shows an upward shift in ξ and the orange line shows a downward shift. The horizontal axis corresponds to the production line dimension.

In the model, the parameter ξ governs the share of embedded intangibles allocated to brand value on the demand side, while $1 - \xi$ corresponds to organizational capital on the supply side. A higher ξ implies that brand value dominates the embedded intangible stock, whereas a lower ξ indicates a greater role for organizational capital. Figure 7 shows that firms derive greater benefits from organizational capital than from brand value when expanding their scope. The primary reason is that organizational capital directly offsets the managerial costs associated with horizontal expansion, thereby sustaining productivity and growth across production lines.

By contrast, brand value does not provide this offsetting capacity. When the share of brand value is high, firms have limited incentive to expand horizontally and predominantly concentrate on a single production line. The associated decline in markups may appear counterintuitive, but it is driven by the reduction in organizational capital, which diminishes managerial productivity. This negative supply-side effect outweighs the positive demand-side gains from brand value. Furthermore, when ξ is high, the concentration of embedded intangibles in brand value raises entry barriers, as potential entrants must match this high intangible stock to compete even on a single production line. Consequently, both competitive threat and the aggregate growth rate decline (see Figure C2). All these effects are reversed when ξ is low: a larger share of organizational capital provides firms with a greater expansion advantage, facilitating scope growth and improving the dynamics of creative destruction and aggregate growth.

5. Counterfactual Analysis

In this section, I conduct a series of counterfactual analyses to isolate the mechanisms driving the results. First, I deactivate the span-of-control constraint. Second, I shut down the accumulation of embedded intangibles. Finally, I shut down both mechanisms jointly. This sequence allows me to isolate the contribution of each mechanism and their interaction to markups, the firm size distribution, competitive threat, and aggregate growth.

Shutting Down the Span-of-Control Constraint. To evaluate the role of the span-of-control constraint, I set $n^{\alpha} = 1$, removing the mechanism by which managerial productivity declines with the number of production lines. The results are presented in Figure 8, where the counterfactual is plotted in green and the baseline in red.

Removing this friction generates a rightward shift in the firm size distribution, with firms operating more production lines relative to the baseline. In the absence of diminishing managerial returns, firms can expand horizontally without incurring rising marginal costs. As a result, markups increase with the number of production lines, reversing the baseline pattern in which diversification reduced markups. Under Bertrand competition, a firm's markup in each production line is constrained by its own marginal cost relative to competitors. In the baseline, span-of-control frictions raise marginal costs with expansion, compressing markups. Eliminating this channel keeps marginal costs low across all lines, allowing firms to sustain higher markups as they expand.

However, the relaxation of managerial constraints reduces competitive threat. As firms grow larger and manage more production lines, they face stronger incentives to invest in embedded intangibles due to increasing returns to scale. The resulting accumulation raises entry barriers and lowers competitive threat. Despite this decline, the aggregate growth rate rises, driven by a shift in the source of growth: while creative destruction diminishes, innovation by incumbent superstar firms increases. The enhanced capacity of multi-product firms to expand and innovate boosts firm-level growth rates, raising aggregate growth above the baseline.

Shutting Down Embedded Intangibles. I next examine the effects of eliminating embedded intangibles by setting $\bar{k} = 1$, so that superstar firms can no longer accumulate brand value or organizational capital.

As shown in Figure 8 (blue line), this produces a leftward shift in the firm size distribution, concentrated on a single production line. Without the ability to offset span-of-control costs through embedded intangible accumulation, expansion remains costly and firms have no incentive to diversify. Markups also decline, as firms lose the pricing advantage conferred by embedded intangibles relative to rivals, though this reduction is modest — the markup effect of embedded intangibles is limited in quantitative terms relative to their effects on entry barriers and growth.

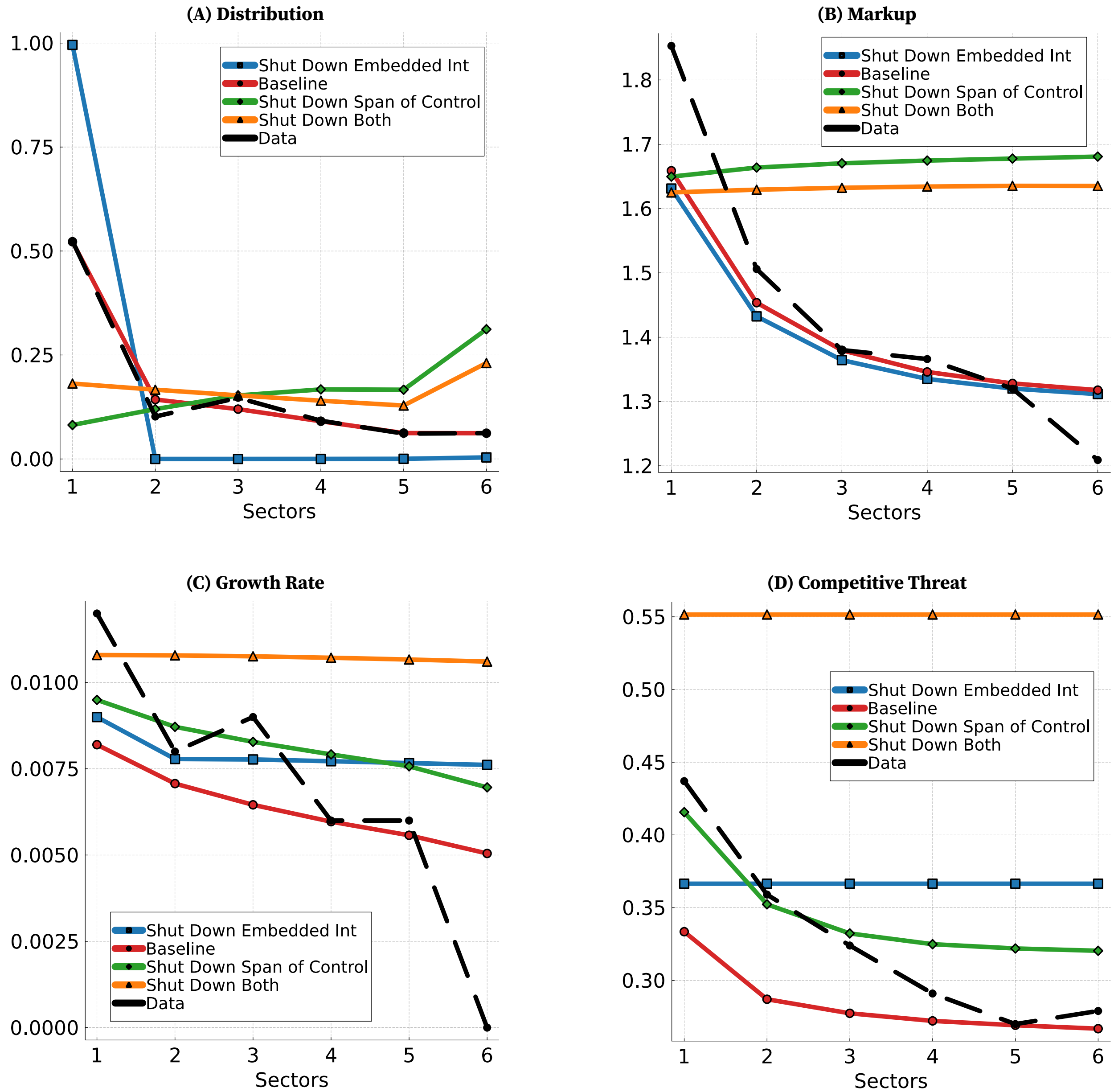

Figure 8. Counterfactual Analysis Under Different Cases

Note: The red line shows the baseline calibration; the green line shows the case with span-of-control constraint shut down ($\alpha_s = 0$); the blue line shows the case with embedded intangibles shut down ($\bar{k} = 1$); and the orange line shows the case with both the span-of-control constraint and embedded intangibles shut down ($\alpha_s = 0$ and $\bar{k} = 1$).

The aggregate growth rate rises, driven primarily by a substantial increase in competitive threat. Eliminating embedded intangibles removes the entry barriers that incumbents build through intangible accumulation, allowing challenger superstars and fringe firms to enter production lines more easily. This increase in creative destruction directly raises the aggregate growth rate, underscoring the central role of embedded intangibles in shaping the dynamics of market competition.

Both counterfactuals raise the aggregate growth rate, but through distinct channels. In the span-of-control counterfactual, the growth increase is driven primarily by incumbent innovation. When embedded intangibles are eliminated, growth is driven instead by innovations from challenger superstars displacing incumbents.

Shutting Down Both Mechanisms. Shutting down both mechanisms simultaneously allows firms to expand horizontally, shifting the firm size distribution rightward. However, the magnitude of this shift is more limited than in the span-of-control counterfactual alone, because firms can no longer leverage the increasing returns to scale from embedded intangibles to facilitate expansion. Markups increase with the number of production lines, but this increase is more muted than in the span-of-control counterfactual, as firms lack the embedded intangible stock that amplifies pricing power with expansion.

By contrast, competitive threat and the aggregate growth rate are higher in this combined counterfactual than in either single-mechanism case, driven by innovation from both incumbent firms and challenger superstars who find entry easier in the absence of embedded intangible barriers.

To summarize, all three counterfactuals establish that both span-of-control constraints and embedded intangibles are necessary to replicate the empirical regularities documented in Section 3.3. Removing the span-of-control constraint causes markups to increase with scope rather than decline, while removing embedded intangibles concentrates the firm size distribution on a single production line and eliminates the declining trends in competitive threat across production lines. Together, these two mechanisms constitute the minimal conditions required to match the key empirical patterns.

6. Policy Implication and Misallocation

This section first quantifies the effects of resource misallocation on aggregate output and economic growth. Based on these insights, I analyze policy tools to examine how these frictions affect welfare and how they can be mitigated.

6.1. Misallocation

Misallocation in the model operates through two distinct channels. The first stems from markup dispersion across firms. The second arises because a potential superstar entrant cannot capture a market unless its embedded intangible level is at least as high as the incumbent's, creating a barrier to entry. To quantify the effect of markup dispersion, I adapt the method by Peters (2020) and decompose aggregate output Y into four components: the contribution from quality improvements (Q), the contribution from embedded intangible capital (E), the misallocation due to markup dispersion (M), and a leftover term (S)[22],

$$Y_t = Q_t \times E_t \times M_t \times S_t \tag{39}$$

$$\text{where} \quad E_t = \exp\left(\sum_m \sum_k \sum_n \frac{1}{\varepsilon} \ln\left[(\xi\theta)^{k\times\beta}\left((1-\xi)\theta)^{k\times\alpha}\right)^{\varepsilon}\right]\mu(m,k,n)\right),$$

$$M_t = \frac{\exp\left(\sum_m \sum_k \sum_n \ln\left[\frac{\phi(m,k,n)}{\sigma(m,k,n)} + (1-\phi(m,k,n)\right]\mu(m,k,n)\right)}{\sum_m \sum_k \sum_n \left(\frac{\phi(m,k,n)}{\sigma(m,k,n)} + (1-\phi(m,k,n)\right)\mu(m,k,n)}, \quad \text{and}$$

$$S_t = \exp\left(\sum_m \sum_k \sum_n \ln\left[\frac{\left(\frac{1}{\gamma n^{\alpha_s}}\frac{\phi(m,k,n)}{\sigma(m,k,n)}\right)^{\varepsilon} + \frac{1}{(\xi\theta^k)^{\beta}}\left(\frac{\lambda^{-m}}{((1-\xi)\theta^k)^{\alpha}}(1-\phi(m,k,n))\right)^{\varepsilon}}{\left(\frac{\phi(m,k,n)}{\sigma(m,k,n)} + (1-\phi(m,k,n))\right)^{\varepsilon}}\right]^{\frac{1}{\varepsilon}}\mu(m,k,n)\right).$$

Defining M as the ratio of the geometric mean to the arithmetic mean, the key misallocation term endogenously evolves with markup dispersion. [23] The results, presented

[22] For details see Appendix B.6

[23] The markup dispersion term M differs slightly from Peters (2020). Even if markups were constant across superstar firms, dispersion between superstars and fringe firms would still exist, governed by the

in Table 3, show that although there is substantial markup dispersion across production lines, the aggregate dispersion effect on misallocation is relatively small. As a result, the misallocation arising purely from markup dispersion is minimal, accounting for only about 0.5% of output.

To analyze the second type of misallocation, I conduct a counterfactual experiment that eliminates entry barriers. In this scenario, any superstar firm that successfully makes a horizontal innovation can immediately enter a new production line and become the incumbent without facing the embedded intangible constraint. The results are striking: aggregate output more than triples. This dramatic gain is primarily driven by a large increase in the quality improvement component (Q). The contribution from embedded intangibles (E) slightly decreases, a likely result of the policy boosting investment in transferable technology (horizontal innovation) and making embedded investment less critical for market entry. Furthermore, the markup dispersion term (M) approaches one, and the residual term (S) increases. These results suggest that policy attention should focus on lowering entry barriers. Enabling frictionless reallocation following horizontal innovations substantially raises aggregate output, primarily through higher quality growth, and therefore can materially reduce misallocation in the economy.

Table 2. Misallocation

	Y	Q	E	M	S	g
Base Scenario	2.512	1.923	0.691	0.995	1.900	0.007
Shut Down Entry Barrier	7.148	4.588	0.673	1.000	2.314	0.017

6.2. Policy Implications

Based on the misallocation channels identified in Section 6.1, I analyze three distinct tax regimes. First, I implement a scope-dependent tax on profits, rising from 10% to 12.5%. Second, I consider a flat tax of 11.3% on investment in embedded intangibles and horizontal expansion. Third, I examine the joint implementation of both tax policies. In all cases,

market share ϕ of superstar firms.

tax revenue is redistributed to households through lump-sum transfers. Welfare effects are evaluated using a consumption-equivalent measure,[24] which quantifies the permanent percentage change in consumption that leaves households indifferent between the baseline and the policy-induced path.

The welfare results show substantial variation across policies. The scope-dependent tax yields a welfare gain of 10.915%, accompanied by increases in consumption and output. This policy operates through two channels. First, it reduces the incentive to expand scope by raising the cost of operating across many production lines, thereby mitigating the span-of-control distortion. Second, it attenuates incentives to over-invest in embedded intangibles by reducing the cross-sector synergies that make such investment attractive, ultimately lowering entry barriers. By contrast, taxes targeting embedded and expansion investments yield more modest individual gains of 1.745% and 2.176%, respectively, as they address the misallocation from intangible accumulation directly but do not simultaneously correct the scope distortion.

The joint implementation of all three taxes produces a welfare gain of 16.237%, exceeding the sum of the individual contributions. This complementarity arises because scope-dependent and intangible-investment taxes address distinct but interacting margins of misallocation: the former corrects the distortion in the horizontal expansion decision, while the latter corrects the distortion in intangible composition. Together they more fully align private and social incentives, generating aggregate efficiency gains that neither policy achieves alone.

Table 3. Welfare and Output Effects at Varying Tax Levels

	Tax Type			
	Scope-Dependent Tax [0.1, 0.105, 0.11, 0.115, 0.12, 0.125]	Embedded Inv Tax 0.113	Expanding Inv Tax 0.113	All Three Combined
Δ Welfare (%)	10.915	1.745	2.176	16.237
ΔC (%)	10.542	1.602	2.093	15.549
ΔY (%)	8.517	1.566	2.084	13.255

[24] See Appendix B.7 for details on the consumption-equivalence welfare calculation.

7. Conclusion

This paper establishes a unified endogenous growth model in which the direction of innovative effort emerges endogenously from firms' expansion decisions. Horizontal expansion introduces span-of-control frictions that reduce managerial quality per sector and lower per-sector profitability. In response, multi-sector firms systematically reallocate investment away from transferable intangibles, which generate knowledge spillovers and fuel creative destruction, and toward embedded intangibles, which provide firm-specific competitive advantages. This reallocation is self-reinforcing: broader scope raises the return to embedded intangibles through economies of scope, inducing further investment in brand value and organizational capital, which in turn lowers the cost of entering additional sectors. As scope and embedded intangible stock grow jointly, the incumbent superstar's total managerial capacity increases, raising entry barriers and suppressing creative destruction — generating a wedge between private and social returns that instruments conditioned on firm size alone cannot correct.

This framework suggests several directions for future research. First, incorporating the evolution of firm scope over the lifecycle would permit a richer analysis of dynamic market segmentation strategies and their long-run consequences for the direction and pace of innovation. Second, a formal comparison between the decentralized equilibrium and a social planner's solution would allow for the precise derivation of optimal policy instruments to address the spillover frictions generated by diversification. Third, the model generates testable predictions regarding how firms respond endogenously to span-of-control constraints — micro-econometric work could examine the relative efficacy of organizational redesign, investment in information technology, and human capital accumulation as adaptive strategies. Finally, a critical open question is whether embedded intangible capital slows idea diffusion by limiting knowledge spillovers. Addressing this with richer microdata and dynamic structural methods will be essential for designing policies that balance the private gains from diversification against the broader social returns to innovation and competition.

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Appendices

A. Empirical Appendix

A.1. Dataset and Measurement Details

Table A1. Example Firms Segment in Compustat Segment Dataset

Company	**Segments**
TOYOTA MOTOR CORP	Financial Services Automotive All Other
PROCTER & GAMBLE CO	Health Care Grooming Corporate Beauty Baby, Feminine & Family Care Fabric & Home Care
TESLA INC	Energy Generation & Storage Automotive

Measurement Details. The sample covers 1990–2019. The finance and utilities sectors are excluded from all analyses, and estimations are conducted separately for each 2-digit NAICS industry. I employ the approach of Gandhi, Navarro, and Rivers (2020)(GNR) to estimate firm-level total factor productivity. The gross-output production function is specified as

$$Y_{it} = F(L_{it}, K_{it}, M_{it}) + \omega_{it} + \epsilon_{it}, \tag{40}$$

where Y_{it} is (log) gross output (`sale`), L_{it} is labor (`emp`), K_{it} is capital (`ppegt`), and M_{it} denotes flexible (intermediate) inputs (`cogs`). The term ω_{it} denotes an unobserved firm productivity shock that is observed by firms when choosing inputs, and ϵ_{it} is an iid error term.

A central identification challenge is that the presence of flexible inputs creates a non-identification problem for nonparametric gross-output production functions when standard proxy-variable approaches are applied, because flexible-input choices reflect contemporaneous productivity. GNR resolve this by exploiting a transformation of the firm's short-run first-order condition for intermediates to obtain cross-equation restrictions that isolate the flexible-input contribution and thus permit nonparametric identification of the production function and input elasticities.

Empirical proxy (material share): Define the intermediate share

$$s_{it} \equiv \frac{\texttt{COGS}_{it}}{\texttt{Sales}_{it}},$$

with both numerator and denominator deflated by `cpi`. GNR show that s_{it} is the empirical moment implied by the transformed FOC and can be used to recover the flexible-input elasticity nonparametrically[25].

First stage (nonparametric share regression): I apply the GNR transformation of the FOC for intermediates and estimate the resulting relation between s_{it} and the observable state variables nonparametrically. This yields an observation-level flexible-input elasticity $\hat{\beta}_{m,it}$

Second stage (fixed-input elasticities and TFP): With $\hat{\beta}_{m,it}$ in hand, the second stage identifies the remaining input elasticities $\hat{\beta}_l, \hat{\beta}_k$ using the cross-equation restrictions and standard Markov term $\omega_{it} = g(\omega_{it-1}) + \xi_{it}$. Construct firm TFP as the residual

$$\hat{\omega}_{it} = Y_{it} - \hat{\beta}_l L_{it} - \hat{\beta}_k K_{it} - \hat{\beta}_{m,it} M_{it}.$$

[25]Estimating using methodology by Ackerberg, Caves, and Frazer (2015) and Levinsohn and Petrin (2003) methodology, I replace the proxy variable "revenue share" with "capital expenditure" (`capx`).

A.2. Additional Figures and Empirical Results

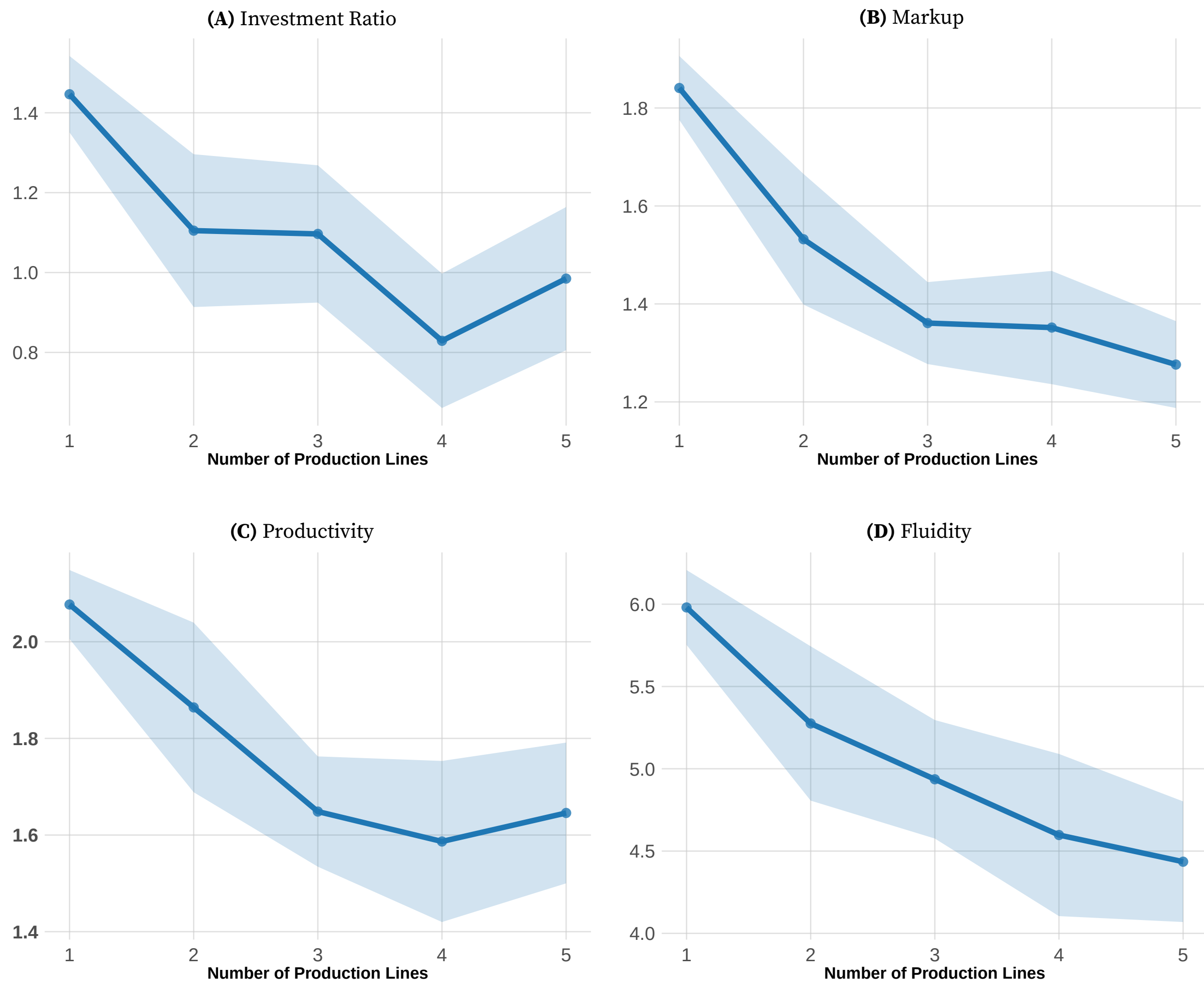

Figure A1. US Firms; Markup, Productivity, Investment Ratio and Fluidity by Production Lines

Note: The sample excludes utilities and finance sectors, as well as firms with missing or non-positive R&D and SG&A. Variables are measured for the 2019 cross-section and winsorized at the 95th percentile.

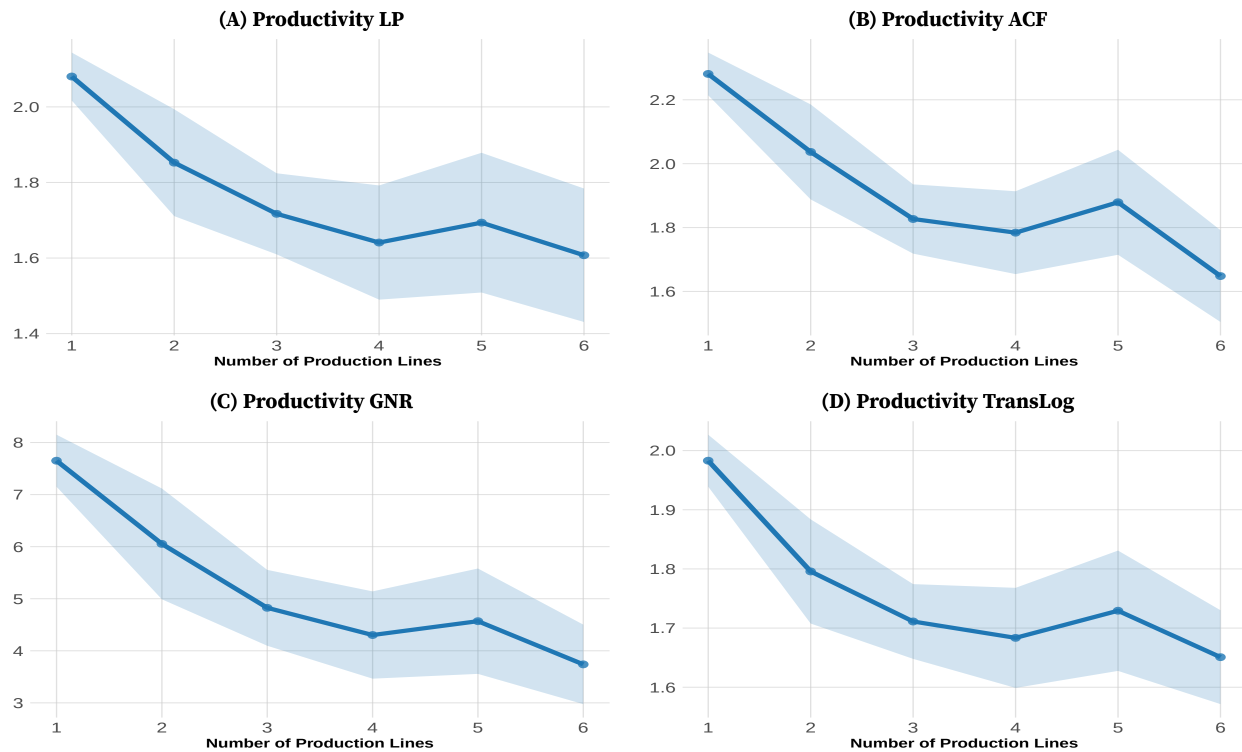

Figure A2. Productivity Measures with Different Methods

Note: The sample excludes utilities and finance sectors, as well as firms with missing or non-positive R&D and SG&A. All variables are winsorized at the 95th percentile. Productivities are measured for the 2019 cross-section. LP: Levinsohn and Petrin (2003); ACF: Ackerberg, Caves, and Frazer (2015); GNR: Gandhi, Navarro, and Rivers (2020)

Table A2. Summary Statistics

Variable	Summary Statistics						
	Mean	SD	Median	P10	P25	P75	P90
Investment ratio	1.300	1.310	0.873	0.129	0.317	1.760	3.290
R&D to Sales ratio	0.143	0.187	0.0715	0.006 93	0.0202	0.185	0.364
Markup	1.630	0.830	1.320	0.870	1.020	1.990	3.430
Productivity	1.910	0.972	1.730	0.754	1.010	2.770	3.300
Log sales	13.100	2.570	13.200	9.840	11.400	14.900	16.400
Log total assets	13.600	2.520	13.600	10.300	11.800	15.300	16.800
Log employees	7.490	2.310	7.530	4.380	5.820	9.130	10.500
Number of unique firms				1,711			

This table reports summary statistics for firm characteristics and the main variables used in the paper. The investment ratio of Transferable over Embedded is defined in Section ??. Investment ratio, R&D-to-sales, and productivity are winsorized at the 95th percentile, while markup is winsorized at the 90th percentile. All other variables are presented in logarithmic form.

Table A3. Regression Results: Compustat Dataset

Panel A: *Markup* and *Productivity*

	Pooled OLS		Two-way FE	
	(2) *Markup*	(3) *Productivity*	(6) *Markup*	(7) *Productivity*
Production Lines	-0.099^{***}	-0.056^{***}	-0.111^{***}	-0.032^{***}
	(0.007)	(0.010)	(0.008)	(0.004)
Num. Obs.	40,517	40,517	40,517	40,517
Adj. R^2	0.042	0.025	0.111	0.808
Covariates	Yes	Yes	Yes	Yes
FE: *Year*	No	No	Yes	Yes
FE: *Industry*	No	No	Yes	Yes

Panel B: *Investment Ratio* and *Fluidity*

	Pooled OLS		Two-way FE	
	(1) *Transferable/Embedded*	(4) *Comp. Threat*	(5) *Transferable/Embedded*	(8) *Comp. Threat*
Production Lines	-0.092^{***}	-0.449^{***}	-0.100^{***}	-0.302^{***}
	(0.008)	(0.031)	(0.009)	(0.033)
Num. Obs.	41,071	30,403	41,071	30,403
Adj. R^2	0.085	0.043	0.127	0.169
Covariates	Yes	Yes	Yes	Yes
FE: *Year*	No	No	Yes	Yes
FE: *Industry*	No	No	Yes	Yes

Notes: Each column reports coefficients from a separate regression. Standard errors are clustered by *firm id* in parentheses. Significance levels: $^{*}p < 0.05$, $^{**}p < 0.01$, $^{***}p < 0.001$. Columns 1–4: Pooled OLS specifications; Columns 5–8: Two-way fixed effects (year and industry).

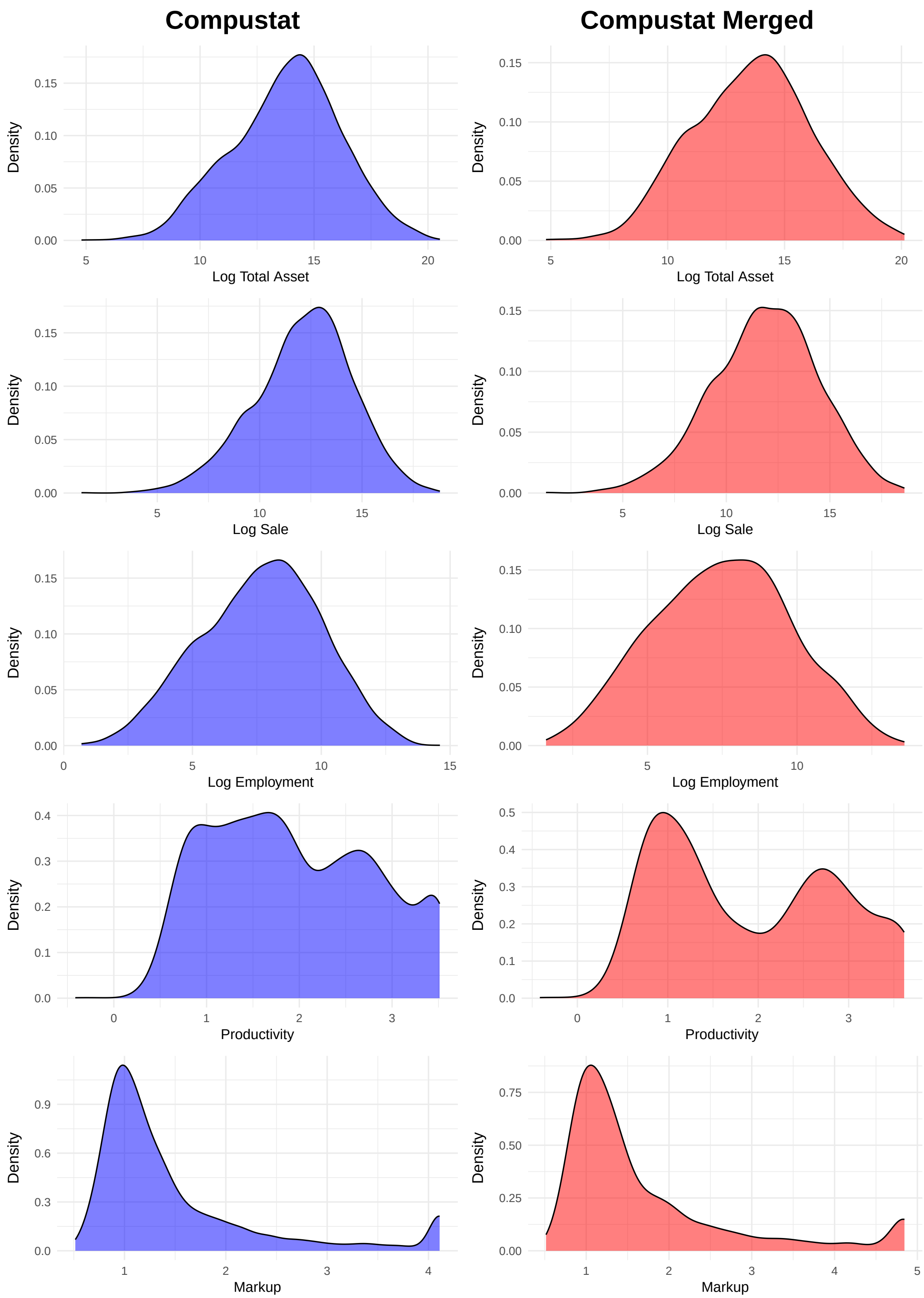

Figure A3. Density Comparison Compustat and Compustat Merged

Note: Both datasets exclude utilities and finance sectors, and all variables are measured for the 2019 cross-section. The Compustat merged dataset further excludes firms with missing or non-positive R&D, SG&A, and segment information.

B. Model Appendix

B.1. Final Good Sector Demand

The final good sector's profit maximization problem is:

$$\max_{y_{jt}} \exp\left(\int_0^1 \ln y_{jt} dj\right) - \int_0^1 p_{jt} y_{jt} dj. \tag{41}$$

The first-order condition yields the inverse demand function for each intermediate good j:

$$p_{jt} = \frac{Y_t}{y_{jt}}. \tag{42}$$

B.2. Intermediate Good Sector Demand

The cost minimization problem for the intermediate sector j is:

$$\min_{y_{sjt}, y_{fjt}} \quad p_{sjt} y_{sjt} + p_{fjt} y_{fjt} \quad s.t. \quad y_{jt} = \left(\chi(e_s) y_{sjt}^{\varepsilon} + y_{fjt}^{\varepsilon}\right)^{\frac{1}{\varepsilon}}. \tag{43}$$

The first-order condition with respect to the superstar firm's output y_{sjt} is:

$$p_{sjt} = \lambda \chi(e_s) y_{jt}^{1-\varepsilon} y_{sjt}^{\varepsilon-1}, \tag{44}$$

where λ is the Lagrange multiplier. Raising both sides to the power of $\frac{\varepsilon}{\varepsilon-1}$ and simplifying allows to solve for the ideal price index p_j for sector j:

$$\lambda = \left(\chi(e_s)^{\frac{-1}{\varepsilon-1}} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} + p_{fjt}^{\frac{\varepsilon}{\varepsilon-1}}\right)^{\frac{\varepsilon-1}{\varepsilon}} \equiv p_j. \tag{45}$$

Substituting $\lambda = p_j$ back into the first-order condition yields the inverse demand function faced by the superstar firm:

$$p_{sjt} = p_j \chi(e_s) y_{jt}^{1-\varepsilon} y_{sjt}^{\varepsilon-1}. \tag{46}$$

Finally, substituting the final good producer's demand $y_{jt} = Y_t/p_j$ and solving for y_{sjt} provides demand function:

$$y_{sjt} = p_j^{\frac{\varepsilon}{1-\varepsilon}} \chi(e_s)^{\frac{1}{1-\varepsilon}} p_{sjt}^{\frac{1}{\varepsilon-1}} Y_t. \tag{47}$$

B.3. Superstar Firm Maximization Problem: Bertrand Competition

The superstar firm competes à la Bertrand with a continuum of fringe firms. Its profit maximization problem in industry j is:

$$\max_{p_{sjt}} (p_{sjt} - MC_{sjt})\, y_{sjt} \quad s.t. \quad y_{sjt} = p_j^{\frac{\varepsilon}{1-\varepsilon}} \chi(e_s)^{\frac{1}{1-\varepsilon}} p_{sjt}^{\frac{1}{\varepsilon-1}} Y_t. \tag{48}$$

Substituting the expression for p_j into the demand function, the objective function can be expanded as:

$$\max_{p_{sjt}} \left[p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} \chi(e_s)^{\frac{1}{1-\varepsilon}} Y_t \left(\chi(e_s)^{\frac{-1}{\varepsilon-1}} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} + p_{fjt}^{\frac{\varepsilon}{\varepsilon-1}} \right)^{-1} - MC_{sjt} \left(\chi(e_s)^{\frac{-1}{\varepsilon-1}} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} + p_{fjt}^{\frac{\varepsilon}{\varepsilon-1}} \right)^{-1} p_{sjt}^{\frac{1}{\varepsilon-1}} \chi(e_s)^{\frac{1}{1-\varepsilon}} Y_t \right]. \tag{49}$$

After computing the derivative and factoring common terms, this condition can be expressed as:

$$\begin{aligned} \frac{\partial \pi}{\partial p_{sjt}} = Y_t \chi(e_s)^{\frac{1}{1-\varepsilon}} \left(\chi(e_s)^{\frac{-1}{\varepsilon-1}} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} + p_{fjt}^{\frac{\varepsilon}{\varepsilon-1}} \right)^{-1} \times \Bigg\{ \left[\frac{\varepsilon}{\varepsilon-1} p_{sjt}^{\frac{1}{\varepsilon-1}} - \text{MC}_{sjt} \frac{1}{\varepsilon-1} p_{sjt}^{\frac{2-\varepsilon}{\varepsilon-1}} \right] \\ - \left(p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} - \text{MC}_{sjt} p_{sjt}^{\frac{1}{\varepsilon-1}} \right) \left(\chi(e_s)^{\frac{-1}{\varepsilon-1}} \frac{\varepsilon}{\varepsilon-1} p_{sjt}^{\frac{1}{\varepsilon-1}} \left(\chi(e_s)^{\frac{-1}{\varepsilon-1}} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} + p_{fjt}^{\frac{\varepsilon}{\varepsilon-1}} \right)^{-1} \right) \Bigg\} = 0. \end{aligned} \tag{50}$$

$$\begin{aligned} 0 = \left[\frac{\varepsilon}{\varepsilon-1} p_{sjt}^{\frac{1}{\varepsilon-1}} - \text{MC}_{sjt} \frac{1}{\varepsilon-1} p_{sjt}^{\frac{2-\varepsilon}{\varepsilon-1}} \right] \\ - \left(p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} - \text{MC}_{sjt} p_{sjt}^{\frac{1}{\varepsilon-1}} \right) \left(\chi(e_s)^{\frac{-1}{\varepsilon-1}} \frac{\varepsilon}{\varepsilon-1} p_{sjt}^{\frac{1}{\varepsilon-1}} \left(\chi(e_s)^{\frac{-1}{\varepsilon-1}} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} + p_{fjt}^{\frac{\varepsilon}{\varepsilon-1}} \right)^{-1} \right). \end{aligned} \tag{51}$$

To simplify, multiply both sides by p_{sjt} and substitute the market share definition ϕ_{sjt} from (19), yielding:

$$\left(p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} - \mathrm{MC}_{sjt} p_{sjt}^{\frac{1}{\varepsilon-1}}\right)\left(\phi_{sjt}\frac{\varepsilon}{\varepsilon-1}\right) = \left[\frac{\varepsilon}{\varepsilon-1} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} - \mathrm{MC}_{sjt}\frac{1}{\varepsilon-1} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}}\right]. \tag{52}$$

First, divide both sides by $p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}}$, then by $\frac{\varepsilon}{\varepsilon-1}$. After rearranging terms, the expression simplifies to:

$$\varepsilon(1-\phi_{sjt}) = \frac{\mathrm{MC}_{sjt}}{p_{sjt}}(1-\varepsilon\phi_{sjt}). \tag{53}$$

Solving for the optimal price p_{sjt} gives:

$$p_{sjt} = \frac{1-\varepsilon\phi_{sjt}}{\varepsilon(1-\phi_{sjt})} \cdot \mathrm{MC}_{sjt}. \tag{54}$$

To determine the optimal labor demand, equating output (6) and demand (47) yields

$$q_{sjt}\,\psi(e_s,n_s)\,l_{sjt} = p_j^{\frac{\varepsilon}{1-\varepsilon}} \chi(e_s)^{\frac{1}{1-\varepsilon}} p_{sjt}^{\frac{1}{\varepsilon-1}} Y_t. \tag{55}$$

Multiplying both sides by p_{sjt} and dividing by w_t gives

$$p_{sjt}\underbrace{\frac{q_{sjt}\,\psi(e_s,n_s)}{w_t}}_{\text{inverse MC}_{sjt}}\,l_{sjt} = \underbrace{p_j^{\frac{\varepsilon}{1-\varepsilon}} \chi(e_s)^{\frac{1}{1-\varepsilon}} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}}}_{\phi_{sjt}}\underbrace{\frac{Y_t}{w_t}}_{\omega_t^{-1}}. \tag{56}$$

This expression leads directly to equation (24).

B.4. Superstar Firm Maximization Problem: Cournot Competition

In the à la Cournot setup, a superstar firm and a continuum of fringe firms compete by choosing quantities to sell rather than engaging in price competition as in the à la Bertrand case. Its profit-maximization problem in industry j is

$$\max_{y_{sjt}} \pi_{sjt} = \max_{y_{sjt}} (p_{sjt} - MC_{sjt})\, y_{sjt}, \tag{57}$$

$$s.t. \quad p_{sjt} = \chi(e_s)\, y_{jt}^{-\varepsilon}\, y_{sjt}^{\varepsilon-1}\, Y_t, \quad \text{and} \quad y_{jt} = \left(\chi(e_s)\, y_{sjt}^{\varepsilon} + y_{fjt}^{\varepsilon}\right)^{1/\varepsilon}. \tag{58}$$

Differentiate the profit function with respect to y_{sjt}:

$$\frac{d\pi_{sjt}}{dy_{sjt}} = (p_{sjt} - MC_{sjt}) + y_{sjt}\, \frac{dp_{sjt}}{dy_{sjt}} = 0. \tag{59}$$

Differentiate the inverse demand (58) to obtain

$$\frac{dp_{sjt}}{dy_{sjt}} = p_{sjt}\left(-\varepsilon \frac{1}{y_{jt}} \frac{dy_{jt}}{dy_{sjt}} + \frac{\varepsilon - 1}{y_{sjt}}\right). \tag{60}$$

Differentiate y_{jt} with respect to y_{sjt}:

$$\frac{dy_{jt}}{dy_{sjt}} = \chi(e_s)\, y_{sjt}^{\varepsilon-1}\, y_{jt}^{1-\varepsilon}. \tag{61}$$

Substituting (61) and (58) into (59) yields

$$(p_{sjt} - MC_{sjt}) + y_{sjt}\, p_{sjt}\left(-\varepsilon\chi(e_s)\, y_{sjt}^{\varepsilon-1} y_{jt}^{-\varepsilon} + \frac{\varepsilon - 1}{y_{sjt}}\right) = 0. \tag{62}$$

Using the market-share definition

$$\frac{p_{sjt}\, y_{sjt}}{Y_t} = \chi(e_s)\, y_{sjt}^{\varepsilon}\, y_{jt}^{-\varepsilon},$$

and dividing both sides by p_{sjt}, rearrangement gives the inverse markup condition

$$\frac{MC_{sjt}}{p_{sjt}} = \varepsilon(1 - \phi_{sjt}). \tag{63}$$

The relative price ratio between fringe and superstar firms is therefore

$$\frac{p_{fjt}}{p_{sjt}} = \frac{1 - \phi_{sjt}}{\phi_{sjt}}\, \frac{MC_{fjt}}{MC_{sjt}}. \tag{64}$$

Using the inverse demand expressions leads to

$$\chi(e_s)\left(\frac{y_{fjt}}{y_{sjt}}\right)^{\varepsilon-1}=\frac{1-\phi_{sjt}}{\phi_{sjt}}\,\frac{MC_{fjt}}{MC_{sjt}}. \tag{65}$$

Substituting the marginal-cost expressions for the fringe and superstar firms yields

$$\left(\frac{y_{fjt}}{y_{sjt}}\right)^{\varepsilon-1}=\frac{1}{\chi(e_s)}\,\frac{1-\phi_{sjt}}{\phi_{sjt}}\,\frac{1}{\lambda^{m_j}\,\psi(e_s,n_s)}. \tag{66}$$

Equation (66) shows that the relative output depends on the market share, the quality gap, the embedded intangible level, and the firm's production-line parameter. Rearranging the market-share definition gives

$$\chi(e_s)\,y_{sjt}^{\varepsilon}\,y_{jt}^{-\varepsilon}=\frac{\chi(e_s)\,y_{sjt}^{\varepsilon}}{\chi(e_s)\,y_{sjt}^{\varepsilon}+y_{fjt}^{\varepsilon}}=\frac{1}{1+\dfrac{1}{\chi(e_s)}\left(\dfrac{y_{fjt}}{y_{sjt}}\right)^{\varepsilon}}, \tag{67}$$

which shows that market share depends on the output gap and the embedded intangible level. Therefore, the output gap depends on the quality gap, the embedded intangible level, and the number of production lines associated with superstar s.

B.5. Aggregate Output and Growth Rate

Using the superstar (6) and fringe firm (7) output equations into (3) gives

$$Y_t=\exp\left(\int_0^1\ln\left[(\xi e_{st})^{\beta}\Big(q_{sjt}\,\frac{((1-\xi)e_{st})^{\alpha}}{\gamma n_{st}^{\alpha_s}}\,l_{sjt}\Big)^{\varepsilon}+\big(q_{fjt}\,l_{fst}\big)^{\varepsilon}\right]^{1/\varepsilon}dj\right)$$

$$Y_t=\exp\left(\int_0^1\ln\left[\left(q_{sjt}^{\varepsilon}\left((\xi e_{st})^{\beta}\Big(\frac{((1-\xi)e_{st})^{\alpha}}{\gamma n_{st}^{\alpha_s}}\,l_{sjt}\Big)^{\varepsilon}+\big(\lambda^{-m_{jt}}\,l_{fst}\big)^{\varepsilon}\right)\right)\right]^{1/\varepsilon}dj\right)$$

$$=\underbrace{\exp\left(\int_0^1\ln q_{sjt}dj\right)}_{Q_t}\exp\left(\int_0^1\ln\left[((\xi e_{st})^{\beta}\left(\frac{((1-\xi)e_{st})^{\alpha}}{\gamma n_{st}^{\alpha_s}}\,l_{sjt}\right)^{\varepsilon}+\left(\lambda^{-m_{jt}}\,l_{fst}\right)^{\varepsilon}\right]^{\frac{1}{\varepsilon}}dj\right), \tag{68}$$

and , along with the labor demands in (24) and (**??**), and factoring out ω_t^{-1}, aggregate output can be expressed as

$$Y_t = Q_t\,\omega_t^{-1}\,\exp\left(\int_0^1 \ln\left[(\xi e_{st})^{\beta}\left(\frac{((1-\xi)e_{st})^{\alpha}}{\gamma n_{st}^{\alpha_s}}\frac{\phi_{sjt}}{\sigma_{sjt}}\right)^{\varepsilon} + \left(\lambda^{-m_{jt}}(1-\phi_{sjt})\right)^{\varepsilon}\right]^{\frac{1}{\varepsilon}} dj\right). \quad (69)$$

Because everything in the integrand depends only on the gaps (m,k,n), the expression can be written in discrete state space as

$$Y_t = Q_t\,\omega_t^{-1}\,\exp\left(\sum_{m=1}^{\bar m}\sum_{k=1}^{\bar k}\sum_{n=1}^{\bar n} \underbrace{\ln\left[(\xi\theta^{k})^{\beta}\left(\frac{((1-\xi)\theta^{kt})^{\alpha}}{\gamma n^{\alpha_s}}\frac{\phi_t(m,k,n)}{\sigma_t(m,k,n)}\right)^{\varepsilon} + \left(\lambda^{-mt}(1-\phi_t(m,k,n))\right)^{\varepsilon}\right]^{\frac{1}{\varepsilon}}}_{\equiv R_t(m,k,n)} \mu_t(m,k,n)\right). \quad (70)$$

$$\begin{aligned}\ln Y_{t+\Delta t} - \ln Y_t = {} & \big(\ln Q_{t+\Delta t} - \ln Q_t\big) + \ln\omega_t - \ln\omega_{t+\Delta t} \\ & + \sum_{m=1}^{\bar m}\sum_{k=1}^{\bar k}\sum_{n=1}^{\bar n}\Big(R_{t+\Delta t}(m,k,n) - R_t(m,k,n)\Big)\Big(\mu_{t+\Delta t}(m,k,n) - \mu_t(m,k,n)\Big) + o(\Delta t).\end{aligned} \quad (71)$$

where

$$\begin{aligned}\ln Q_{t+\Delta t} - \ln Q_t = {} & \ln\lambda\left[\sum_{m=1}^{\bar m}\sum_{k=1}^{\bar k}\sum_{n=1}^{\bar n}\Big(z_t^{Int}(m,k,n) + p^{\mathrm{Ex}}_{s\geq s'}\,z_t^{Ex}(m,k,n) + Z_t^{f}(m,k,n)\Big)\mu_t(m,k,n)\right]\Delta t \\ & + o(\Delta t).\end{aligned} \quad (72)$$

Dividing by Δt and taking the limit $\Delta t \to 0$, the growth rate of the economy is

$$g_t = -g_{\omega,t} + g_{Q,t} + g_{R,t}. \quad (73)$$

In the steady state, the distribution $\mu_t(m,k,n)$ is constant, implying that R_t is constant. Wages grow at the same rate as output, so the real wage remains constant. Therefore, in steady state the growth rate of the economy is determined solely by quality improvements:

$$g = g_Q = \ln \lambda \left[\sum_{m=1}^{\bar{m}} \sum_{k=1}^{\bar{k}} \sum_{n=1}^{\bar{n}} \Big(z^{\text{Int}}(m,k,n) + p^{\text{Ex}}_{s \geq s'}\, z^{\text{Ex}}(m,k,n) + Z^f(m,k,n) \Big) \mu(m,k,n) \right]. \tag{74}$$

B.6. Decomposition of Output

Starting from (69) and factoring out the term $(\xi e_{st})^{\beta}\left(\frac{((1-\xi)e_{st})^{\alpha}}{\gamma\, n_{st}^{\alpha_s}}\right)$, aggregate output can be written as

$$Y_t = Q_t\, \omega_t^{-1} \exp\left(\int_0^1 \ln \left[\left(\frac{\phi_{sjt}}{\sigma_{sjt}}\right)^{\varepsilon} + \frac{1}{(\xi e_{st})^{\beta}} \left(\frac{\gamma\, n_{st}^{\alpha_s}}{((1-\xi)e_{st})^{\alpha}}\, \lambda^{-m_{jt}}(1-\phi_{sjt}) \right)^{\varepsilon} \right]^{1/\varepsilon} dj \right). \tag{75}$$

Define the multiplicative factor that collects the factored-out terms as

$$E_t \;=\; \exp\left(\int_0^1 \ln \left[(\xi e_{st})^{\beta} \left(\frac{((1-\xi)e_{st})^{\alpha}}{\gamma\, n_{st}^{\alpha_s}} \right)^{\varepsilon} \right]^{1/\varepsilon} dj \right). \tag{76}$$

Next multiply and divide the integrand by the linear weight $\dfrac{\phi_{sjt}}{\sigma_{sjt}} + (1-\phi_{sjt})$. After this algebraic step I obtain a decomposition that isolates a simple mean term and a residual term:

$$\begin{aligned} Y_t = Q_t\, E_t\, \omega_t^{-1} \exp\left(\int_0^1 \ln \left[\frac{\phi_{sjt}}{\sigma_{sjt}} + (1-\phi_{sjt}) \right] dj \right) \\ \times \exp\left(\int_0^1 \ln \left[\frac{ \left(\frac{1}{\gamma n_{st}^{\alpha_s}} \frac{\phi_{sjt}}{\sigma_{sjt}} \right)^{\varepsilon} + \frac{1}{(\xi e_{st})^{\beta}} \left(\frac{1}{((1-\xi)e_{st})^{\alpha}} \lambda^{-m_{jt}}(1-\phi_{sjt}) \right)^{\varepsilon} }{ \frac{\phi_{sjt}}{\sigma_{sjt}} + (1-\phi_{sjt}) } \right]^{1/\varepsilon} dj \right). \end{aligned} \tag{77}$$

Finally, using (34) and defining the multiplicative mean term

$$M_t \;=\; \frac{ \exp\Big(\int_0^1 \ln \left[\frac{\phi_{sjt}}{\sigma_{sjt}} + (1-\phi_{sjt}) \right] dj \Big) }{ \int_0^1 \left[\frac{\phi_{sjt}}{\sigma_{sjt}} + (1-\phi_{sjt}) \right] dj }, \tag{78}$$

the output decomposition can be written as

$$Y_t \;=\; Q_t \times E_t \times M_t \times S_t,$$

where

$$S_t = \exp\left(\int_0^1 \ln\left[\frac{\left(\dfrac{1}{\gamma\, n_{st}^{\alpha_s}}\dfrac{\phi_{sjt}}{\sigma_{sjt}}\right)^{\varepsilon} + \dfrac{1}{(\xi e_{st})^{\beta}}\left(\dfrac{1}{((1-\xi)e_{st})^{\alpha}}\,\lambda^{-m_{jt}}(1-\phi_{sjt})\right)^{\varepsilon}}{\dfrac{\phi_{sjt}}{\sigma_{sjt}} + (1-\phi_{sjt})}\right]^{1/\varepsilon} dj\right). \tag{79}$$

B.7. Consumption Equivalence Welfare Measure

On the balanced growth path, consumption grows at rate g, so that $C(t) = C_0 \exp(gt)$.[26] Defining welfare Ω as the present value of lifetime utility from consumption yields:

$$\Omega = \int_0^{\infty} e^{-\rho t} \ln(C(t))\mathrm{d}t \tag{81}$$

$$= \ln(C_0)\int_0^{\infty} e^{-\rho t}\mathrm{d}t + g\int_0^{\infty} te^{-\rho t}\mathrm{d}t. \tag{82}$$

Solving these integrals gives:

$$\Omega = \frac{\ln(C_0)}{\rho} + \frac{g}{\rho^2} \tag{83}$$

$$= \frac{1}{\rho}\left(\ln C_0 + \frac{g}{\rho}\right). \tag{84}$$

Equivalent Welfare Changes Between Economies To compare welfare between two economies—a calibrated benchmark economy (Cal) and a taxed economy (Tax) on their respective balanced growth paths—I compute the percentage change δ in lifetime consumption that

[26]The C_0 consumption level is given by:

$$C_0 = Y_0 - \int_0^1 \left(I_0^{\mathrm{Int}} + I_0^{\mathrm{Emb}} + I_0^{\mathrm{Ex}} + I_0^{f}\right) dj + G_0. \tag{80}$$

In this equation, the subscript 0 represents calibrated optimum values on the balanced growth path, and G is the lump-sum transfer of government taxes.

would make households indifferent between the two. The required compensation δ satisfies:

$$\Omega^{\text{Tax}} = \frac{1}{\rho}\left(\ln\left(C_0^{\text{Cal}}(1+\delta)\right) + \frac{g^{\text{Cal}}}{\rho}\right). \tag{85}$$

Solving equation (85) for δ:

$$\frac{\ln C_0^{\text{Tax}}}{\rho} + \frac{g^{\text{Tax}}}{\rho^2} = \frac{\ln[C_0^{\text{Cal}}(1+\delta)]}{\rho} + \frac{g^{\text{Cal}}}{\rho^2} \tag{86}$$

$$\ln\left(\frac{C_0^{\text{Tax}}}{C_0^{\text{Cal}}}\right) + \frac{g^{\text{Tax}} - g^{\text{Cal}}}{\rho} = \ln(1+\delta) \tag{87}$$

$$\delta = \frac{C_0^{\text{Tax}}}{C_0^{\text{Cal}}} \exp\left(\frac{g^{\text{Tax}} - g^{\text{Cal}}}{\rho}\right) - 1. \tag{88}$$

If $\delta > 0$: households require compensation to remain in the benchmark economy (Cal).
If $\delta < 0$: households would pay to move to the taxed economy (Tax).

C. Numerical Appendix

C.1. Additional Numerical Results

Table C1. Sensitivity Matrix

Parameter	Markup	Growth Rate	Investment Ratio	Innovation Rate
ε	-1.052%	-2.113%	-25.815%	-3.763%
α	-0.010%	-0.324%	-3.736%	-0.521%
α_s	0.000%	-0.027%	-0.043%	-0.062%
β	-0.002%	-0.064%	-0.862%	-0.106%
γ	-0.013%	-0.398%	-4.342%	-0.639%
ξ	-0.004%	-0.129%	-1.484%	-0.210%

Note: Each row reports the percentage change in variables resulting from a 1% change in the parameter value.

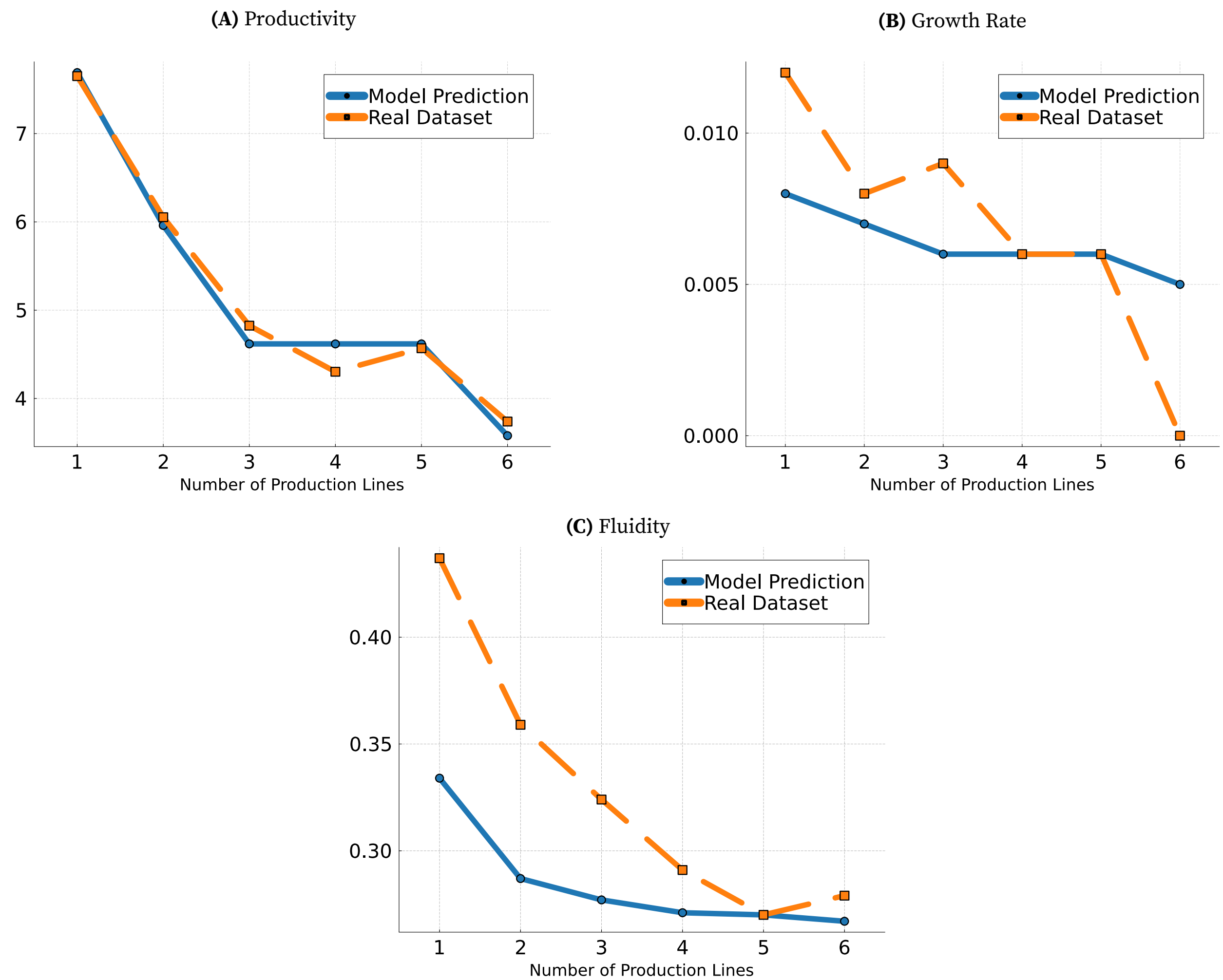

Figure C1. Untargeted Moments: Productivity, Fluidity and Growth Rate by Production Lines

Note: The orange line represents the dataset values, while the blue line shows the model simulation results along the balanced growth path. The horizontal axis corresponds to the production line dimension.

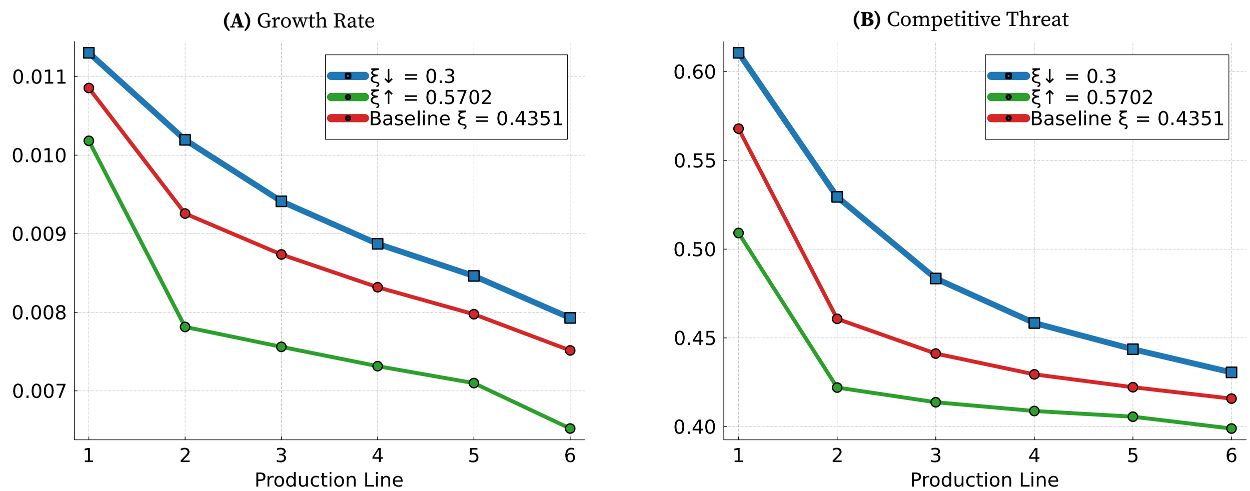

Figure C2. Impact of ξ on Growth Rate and Competitive Threat

Note: The green line represents the internally calibrated optimal value of ξ, while the blue line shows an upward shift in ξ and the orange line shows a downward shift. The horizontal axis corresponds to the production line dimension.

C.2. Empirical counterpart

For scope level n, I aggregate the joint distribution over the quality gap and the embedded-intangible level, described as

$$\mu(n) \;=\; \sum_{m}\sum_{k}\mu(m,k,n).$$

Competitive threat with scope level n is measured as the flow rate of incumbent replacement. It is given by[27]

$$\text{Competitive Threat}(n) = \frac{\sum_{m}\sum_{k'>k}\mathbb{I}\left\{\lambda(\theta^{k'})^{\alpha}n_{s't}^{1-\alpha} > (\theta^{k})^{\alpha}n_{st}^{1-\alpha}\right\} Z^{\text{Hor}}(m,k',n)\mu(m,k',n)}{\mu(n)} + \frac{\sum_{m}\sum_{k}\mathbb{I}_{f>s}Z^{f}(m,k,n)\mu(m,k,n)}{\mu(n)} \tag{89}$$

[27]The first term in the numerator captures horizontal innovations (from higher embedded levels $k' > k$) that displace incumbents in (m,k,n); the second term captures innovation coming from the fringe.

The average markup and the investment ratio at production line n are

$$\sigma(n) = \frac{\sum_m \sum_k \frac{1-\varepsilon\,\phi(m,k,n)}{(1-\phi(m,k,n))\,\varepsilon}\,\mu(m,k,n)}{\mu(n)}, \tag{90}$$

$$\frac{I^T}{I^{\text{Emb}}}(n) = \frac{\sum_m \sum_k \frac{I^{\text{Ver}}(m,k,n)+I^{\text{Hor}}(m,k,n)+I^f(m,k,n)}{I^{\text{Emb}}(m,k,n)}\,\mu(m,k,n)}{\mu(n)}. \tag{91}$$

The growth and productivity by production line are

$$g(n) = \frac{\ln\lambda \sum_m \sum_k \Big(z^{\text{Ver}}(m,k,n) + \mathbb{P}_{k\geq k'}\, z^{\text{Hor}}(m,k,n) + Z^f(m,k,n)\Big)\,\mu(m,k,n)}{\mu(n)}, \tag{92}$$

$$Q(n) = e^{g(n)}. \tag{93}$$

C.3. Solution Algorithm

This algorithm computes the balanced growth path with a three dimensional state space (m,k,n). The solution involves finding the value functions $v_s(m,k,n)$ and $v_f(m,k,n)$, the innovation rates z^{Int}, z^{Emb}, z^{Ex}, z^f, and the stationary distribution $\mu(m,k,n)$ that jointly satisfy the model's equilibrium conditions.

BGP Equilibrium Solution:

1. **Compute static values:** Calculate static market shares and profit values using equations (**??**) and (19).
2. **Initialization:** Initialize the value functions $v_s(m,k,n)$ and $v_f(m,k,n)$, and the stationary distribution $\mu(m,k,n)$.
3. **Step 1: Solve HJB Equations (Backward Iteration)**
   (a) Set $v^{\text{old}}(m,k,n)$.
   (b) Repeat until $\max|v^{\text{new}} - v^{\text{old}}| <$ tolerance:
      - Compute policy functions $x(m,k,n)$ from the FOCs using v^{old}.

- Solve the discretized HJB equations for $v^{\text{new}}(m, k, n)$.
- Update $v^{\text{old}} \leftarrow v^{\text{new}}$.

4. **Step 2: Solve the Kolmogorov Forward Equation (KFE)**
   (a) Set $\mu^{\text{old}}(m, k, n)$.
   (b) Repeat until $\max|\mu^{\text{new}} - \mu^{\text{old}}| <$ tolerance:
      - Solve the discretized KFE for $\mu^{\text{new}}(m, k, n)$ using the policy functions $z^{\text{Int}}, z^{\text{Emb}}, z^{\text{Ex}}, z^{f}$.
      - Update $\mu^{\text{old}} \leftarrow \mu^{\text{new}}$.
5. **Step 3: Repeat Steps Until Value Functions and Distribution Converge**

Finally, to determine the optimal parameter values, search over the parameter space to minimize the objective function,

$$\text{Minimize}(z) = \sum_{z=1}^{Z} \frac{\left|\text{model}(z) - \text{data}(z)\right|}{\frac{1}{2}\left|\text{model}(z)\right| + \frac{1}{2}\left|\text{data}(z)\right|}.$$