Firm Scope and Innovation: The Role of Intangibles*

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Abstract

This paper examines how the composition of intangible investment and the direction of innovation jointly shape long-run growth. I develop an endogenous growth model where firms choose between vertical and horizontal innovation while allocating investment across R&D and firm-specific intangibles—brand value and organizational capital. The model is disciplined by a set of firm-level empirical regularities documenting how productivity, markups, and intangible composition vary systematically with firm scope (the number of sectors a firm operates in). As firms expand across sectors, they reallocate investment from knowledge capital toward firm-specific intangibles. This shift endogenously raises entry barriers and suppresses creative destruction. The calibrated model shows that standard size-based policies are inefficient because they fail to distinguish between vertical and horizontal innovation. In contrast, scope-dependent policies more effectively address intangible-driven externalities, leading to higher rates of creative destruction and long-run growth.

Keywords: Schumpeterian growth, step-by-step innovation, intangibles, firm dynamics, span of control.

JEL Classification: E22, O31, O32, O33, O34.

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1. Introduction

Horizontal and vertical innovation are two core strategies in the modern endogenous growth literature for generating long-run growth. Firms can engage in horizontal innovation by expanding their product portfolios (Klette and Kortum, 2004) or pursue vertical innovation by improving the quality of existing products (Aghion and Howitt, 1992; Aghion *et al.*, 2001). At the same time, the literature also documents that firms invest in intangibles not only to innovate but also to build intangible assets such as brand value and organizational capital that strengthen market positions rather than drive innovation-led growth (Cavenaile and Roldan-Blanco, 2021; Crouzet and Eberly, 2019).

While the directions of innovation and composition of intangibles are well documented separately, understanding their interaction is crucial for innovation policy. To be concrete, Apple is a vertically integrated firm focused on mobile communication technologies and invests heavily in R&D. By contrast, 3M is active in transportation, safety, consumer, and health care industries, and although R&D remains its largest intangible investment, it allocates a substantially larger share toward brand and organizational capital relative to Apple [1]. This contrast in firm scope shows weakness of flat and size-based policies. By treating Apple and 3M symmetrically, such policies can either penalize firms that invest heavily in R&D or equally reward those with excessively accumulated firm-specific intangibles, regardless of their aggregate consequences. Yet existing theories provide no framework for distinguishing firms' investment incentives and direction of innovation together, or for designing policies that account for such differences.

What are the aggregate consequences of the interaction between innovation direction and intangible composition for economic growth and creative destruction? To answer this question, I proceed in two steps. First, I develop a unified theoretical framework in which firms engage in both vertical and horizontal innovation and allocate investment across different types of intangibles. Second, I document a new set of empirical regularities showing how firm-level outcomes and investment choices vary systematically

[1] The measurement details are given in Section 3.2.

with firm scope, which I use to discipline the model. The calibrated model shows that narrow-scope[2] firms concentrate investment in knowledge capital, whereas broad-scope firms allocate relatively more to brand value and organizational capital. This heterogeneity in intangible composition in turn determines firms' innovation direction. This joint determination of innovation direction and intangible composition implies that policies conditioned solely on firm size are insufficient, as firm size conflates vertical and horizontal dimensions and therefore misrepresents firms' intangible investment incentives. By contrast, scope-dependent policies that target the externalities associated with intangible heterogeneity lead to higher rates of creative destruction and innovation.

To formalize intangible heterogeneity, I classify intangible assets according to their spillover effects across firms. In this paper, transferable intangibles consist of knowledge capital[3] with the associated R&D investments. These assets can be transferred between firms and generate spillover effects due to their non-rivalrous and partially excludable nature (Romer, 1990). In contrast, embedded intangibles comprise brand value[4] and organizational capital[5] are inherently firm-specific and inseparable from the firm that created them and do not generate spillovers. Consequently, when a firm exits the market, the economic value of embedded intangibles becomes a sunk cost.

In Section 2, I develop an endogenous growth model that builds on the step-by-step vertical innovation framework of Akcigit and Ates (2023) and the vertical and horizontal innovation framework of Peters (2020). The economy consists of a single final-good sector and a continuum of intermediate-good sectors. In each intermediate-good sector,

[2] Throughout the paper, narrow-scope firms refer to firms active in a single sector, whereas broad-scope firms operate across multiple sectors.

[3] Following Griliches (1979), R&D investment generates a stock of knowledge capital that accumulates over time. Examples include patents and innovation-related software, which embody a firm's technological know-how.

[4] Brand value is a demand shifter, positively influencing the perceived quality of a firm's output (Cavenaile and Roldan-Blanco, 2021; Cavenaile *et al.*, 2025a). Evidence also suggests that brand value increases consumer awareness through targeted marketing (Cavenaile *et al.*, 2025b; Baslandze *et al.*, 2023)

[5] In this paper, organizational capital is conceptualized as managerial productivity, including the firm's embodied managerial talent and its contribution to future production profitability (Carlin *et al.*, 2012; Eisfeldt and Papanikolaou, 2013; Prescott and Visscher, 1980).

a single superstar firm and a continuum of fringe firms produce intermediate goods for the final-good sector under Bertrand competition. Superstar firms can invest in embedded intangibles, which improve the perceived quality of their products and managerial productivity through brand value and organizational capital, respectively. Alternatively, they can invest in transferable intangibles, which enhance product quality in two ways: (i) vertically improving existing products and (ii) expanding their scope through entry into new sectors. Fringe firms, by contrast, produce homogeneous products and cannot accumulate brand value or organizational capital. When a superstar firm exits a sector, its transferable intangibles — the accumulated knowledge in that sector — become freely available to fringe firms, and they can only become new superstars through radical innovation.

The model incorporates two key frictions. First, the two types of intangibles differ in their scalability. Embedded intangibles are fully mobile across sectors, generating economies of scope as their benefits increase with the number of sectors a firm operates in. Transferable intangibles, by contrast, require separate investment in each sector. Second, expanding scope divides managerial attention across sectors, thereby reducing managerial productivity per sector, while vertical innovation imposes no such span-of-control constraint (Lucas, 1978)[6]. At the same time, as a superstar firm expands into more sectors, its total managerial capacity increases even as managerial productivity per sector declines. Superstar firms can further strengthen this capacity through the accumulation of organizational capital. Thus, superstar firms leverage their scope and organizational capital both to enter new sectors more readily and to create entry barriers when facing competitive threats.

In Section 3, I combine firm-level data from Compustat with the product-market fluidity dataset of Hoberg *et al.* (2014) to document how markups, productivity, intangible investment, and competitive pressure vary systematically with firm scope. I use these cross-sectional moments to discipline the calibration of the model. To shed light on the di-

[6]See also Jovanovic (2025). Alternatively, Acemoglu *et al.* (2018) propose that when a firm expands its scope, more skilled labor needs to be allocated to operational activities.

rection and magnitude of these relationships, I employ local projections using the patent-value dataset of (Kogan *et al.*, 2017) as an innovation shock. Single-sector firms show stronger reallocation toward transferable intangibles, larger productivity gains, higher markup responses, and greater competitive pressure from rivals compared to multi-sector firms facing an equivalent shock.

The calibrated model delivers two quantitative findings (Section 4). First, firm scope is the primary determinant of innovation direction: as firms expand into additional sectors, the span-of-control constraint reduces market share per sector, thereby lowering broad-scope firms' incentives for horizontal innovation relative to narrow-scope firms. Conditional on scope, however, intangible composition generates substantial heterogeneity in innovation strategies across firms. Second, the joint expansion of scope and embedded intangibles suppresses creative destruction, driving a wedge between private and social returns to innovation: broad-scope firms leverage both their scope and the higher returns to embedded intangibles through economies of scope to erect larger entry barriers and generate smaller knowledge spillovers, while narrow-scope firms concentrating on transferable intangibles contribute more to aggregate productivity growth.

A fundamental tension in innovation policy is that scope and embedded entry barriers protect incumbents against different competitors (Section 5). The scope barrier primarily deters fringe entry: removing it expands fringe radical innovation but erodes the appropriability that sustains incumbent horizontal investment, and superstar innovation contracts. The embedded barrier primarily deters displacement by rival superstars: removing it accelerates incumbent horizontal innovation but crowds out fringe entry. Each barrier removal reallocates innovation between superstars and the fringe rather than raising aggregate innovation. Embedded intangibles add a further layer to this tension. Although complementary to vertical and horizontal innovation at the firm level, their firm-specific character generates no knowledge spillovers, and their accumulation mechanically dampens aggregate growth on the balanced growth path. Uniform taxation faces the same constraint (Section 6): a scope tax applied to all incumbents compresses every superstar innovation margin simultaneously and lowers the aggregate growth rate by approximately 18 percent relative to baseline. Welfare-improving designs must instead be conditioned on scope. A progressive instrument that taxes multi-sector incumbents while subsidizing vertical innovation in single-sector firms and fringe entry at all scope tiers raises the balanced-growth rate by 27 percent and yields a 14 percent consumption-equivalent welfare gain.

Related Literature. A growing literature documents that intangible investment raises market concentration and markups (Chiavari and Goraya, 2025; Crouzet and Eberly, 2019; Weiss, 2020). Several papers examine how different forms of intangible capital shape innovation and market structure. Cavenaile and Roldan-Blanco (2021) and Cavenaile *et al.* (2025a) show that advertising substitutes for R&D and dampens innovation intensity. Pearce and Wu (2024) argue that brand value can be transferred through secondary markets rather than accumulated solely within the firm, and De Ridder (2024) shows that software acts as an entry barrier by simultaneously raising fixed costs and reducing marginal costs, deters entry. The closest framework is Cavenaile and Roldan-Blanco (2021), in which firm-size-dependent advertising expenditure substitutes for R&D. I depart from their setup by modeling advertising as complementary to R&D, introducing organizational capital, and showing that firm scope governs the composition of intangibles.

The direction of firm innovation has been predominantly explained by firm size. Akcigit and Kerr (2018) document that smaller firms favour external innovation while larger firms pursue quality-improving vertical innovation, and Garcia-Macia *et al.* (2019) shows that most within-firm innovation arises from incumbents improving existing products rather than entering new lines. Berlingieri *et al.* (2025) further documents that firm growth is driven primarily by the rapid addition of products to firms' portfolios. These frameworks treat firm size as the relevant heterogeneity dimension and overlook how intangible composition and firm scope shape innovation direction independently of size.

Resource misallocation is an important source of aggregate productivity losses (Hsieh and Klenow, 2009; Restuccia and Rogerson, 2008), with markup dispersion emerging as an important channel (Peters, 2020; Edmond *et al.*, 2023). A separate literature documents secular declines in knowledge spillovers and business dynamism (Akcigit and Ates, 2021;

Akcigit and Ates, 2023), lower patent quality (Olmstead-Rumsey, 2019), production lock-in (Casal, 2024), and rising entry barriers (Gutiérrez and Philippon, 2019). Aghion *et al.* (2023) provide a unified theoretical framework rationalising these patterns through declining overhead costs and a widening incumbent efficiency advantage. I offer a complementary mechanism: as firm scope expands, investment shifts from transferable toward embedded intangibles, endogenously depressing spillovers and raising entry barriers, a misallocation channel that markup-based measures systematically understate.

How managerial capacity shapes firm size has been studied since Lucas (1978) formalised span-of-control constraints in a model where managerial ability determines the firm-size distribution. Rosen (1982) extends this framework to hierarchical organisations, showing how supervision layers amplify earnings differences across talent levels, and Garicano (2000) formalises the role of knowledge hierarchies in production. Empirical work documents the dispersion of management practices and the limits of managerial attention across firms (Bloom and Van Reenen, 2007; Bandiera *et al.*, 2014; Smeets *et al.*, 2019), and Bloom *et al.* (2014) show that enterprise software directly relaxes these constraints. I incorporate span-of-control frictions that arise from scope expansion into firm dynamics, highlighting a distinct channel through which managerial attention shapes firm growth.

2. Theoretical Model

2.1. Economic Environment

Time is continuous and indexed by t. Household preferences are described by a logarithmic utility function

$$\int_0^\infty e^{-\rho t} \ln(C_t)\, dt, \tag{1}$$

where C_t denotes household consumption and $\rho > 0$ is the time discount rate. The household budget constraint is

$$\dot{A}_t = r_t\, A_t + w_t - C_t, \tag{2}$$

with A_t denoting total household assets, w_t the wage rate, and r_t the interest rate. Labor is inelastically supplied with unit measure, and the consumption good serves as the numeraire. Since households own all firms in the economy, total assets equal the aggregate value of superstar and fringe firms across all intermediate-good sectors,

$$A_t = \int_0^1 \left(V_{sjt} + V_{fjt}\right) dj,$$

where V_{sjt} and V_{fjt} denote the values of the superstar and fringe firms.

The final good is produced by aggregating a continuum of intermediate varieties $j \in [0, 1]$ according to

$$\ln(Y_t) = \int_0^1 \ln(y_{jt})\, dj, \tag{3}$$

in a perfectly competitive market. In each intermediate-good sector, a single superstar firm and a continuum of fringe firms compete à la Bertrand to supply the final good producer. A superstar firm s operates across a set of sectors $J_s \subseteq [0, 1]$, with $n_s = |J_s| \in \mathbb{Z}_+$ denoting the number of sectors in which it holds the leading technology. Output in each sector is aggregated with a constant elasticity of substitution technology

$$y_{jt} = \left(\Xi(e_{st})\, y_{sjt}^{\varepsilon} + \left(1 - \Xi(e_{st})\right) y_{fjt}^{\varepsilon}\right)^{\frac{1}{\varepsilon}} \tag{4}$$

where $\Xi(e_{st}) \equiv \dfrac{\chi(e_{st})}{1 + \chi(e_{st})}$ denotes the perceived quality of superstar firms, and $\varepsilon \in (0, 1)$. In each sector, the superstar firm produces a differentiated good whose perceived quality is increasing in its brand value. Specifically, $\chi(e_{st}) = (\xi e_{st})^{\beta}$ is an endogenous and concave demand shifter, where e_{st} denotes the embedded intangibles of superstar firm s, $\xi \in (0, 1)$ is the share of embedded intangibles associated with brand value, and $\beta \in (0, 1)$ governs the curvature of the demand shifter. The term $\Xi(e_{st})$ captures the notion that a higher stock of brand value raises the perceived quality of the superstar's product relative to the fringe. For simplicity, the brand value of fringe firms is normalized to one, and they produce a homogeneous good aggregated as

$$y_{fjt} = \int_0^1 y_{ijt}\, di, \tag{5}$$

where each fringe firm $i \in (0,1)$ takes the market price as given.

The production function of superstar firm s in sector j at time t is

$$y_{sjt} = \underbrace{q_{sjt}}_{\text{Product Quality}} \underbrace{\psi(e_{st}, n_{st})}_{\text{Managerial Productivity}} \underbrace{l_{sjt}}_{\text{Labor Input}}, \tag{6}$$

where q_{sjt} denotes product quality, l_{sjt} is labor input, and $\psi(e_{st}, n_{st})$ captures managerial productivity, defined as

$$\psi(e_{st}, n_{st}) = \frac{\big((1-\xi)\, e_{st}\big)^{\alpha}}{\eta\, n_{st}^{\alpha}}.$$

The component $(1-\xi)e_{st}$ represents the share of embedded intangibles allocated to organizational capital, which governs managerial efficiency. The variable n_{st} denotes the number of sectors operated by firm s at time t. As firm scope n_{st} increases, managerial productivity per sector declines due to the span-of-control friction, since the firm's managerial attention is distributed across a larger number of sectors, reducing the managerial capacity allocated to each individual sector. The parameter α controls the curvature of both organizational capital and the span-of-control constraint, while η is a scale parameter. Aggregating the managerial productivity of superstar firm s across all active sectors generates total managerial capacity[7]

$$\Lambda \cdot e_{st}^{\alpha} \cdot n_{st}^{1-\alpha}, \quad \text{with} \quad \Lambda \equiv \frac{(1-\xi)^{\alpha}}{\eta}.$$

When $0 < \alpha < 1$, total managerial capacity increases with both organizational capital and firm scope, yet each exhibits diminishing marginal returns.[8] Total managerial ca-

[7]Managerial productivity is symmetric across sectors, so total managerial capacity equals per-sector managerial productivity $\psi(e_{st}, n_{st})$ multiplied by the number of active sectors n_{st}.

[8]When $\alpha > 1$, diseconomies of scope emerge, as coordination costs outweigh the benefits of expansion and total managerial capacity decreases with firm scope.

pacity determines the incumbent superstar's ability to defend its market position against challenger entry, a mechanism formalized in the creative destruction section.

Fringe firms produce output using a linear technology

$$y_{fjt} = q_{fjt} l_{fjt}, \tag{7}$$

where q_{fjt} denotes product quality and l_{fjt} is labor input.[9] They operate in a single sector, cannot accumulate embedded intangibles, and have managerial quality normalized to one. Their product quality is inherited, in the sense that when a superstar firm exits a sector, its transferable intangibles become publicly available and fringe firms adopt the previous superstar's quality level.

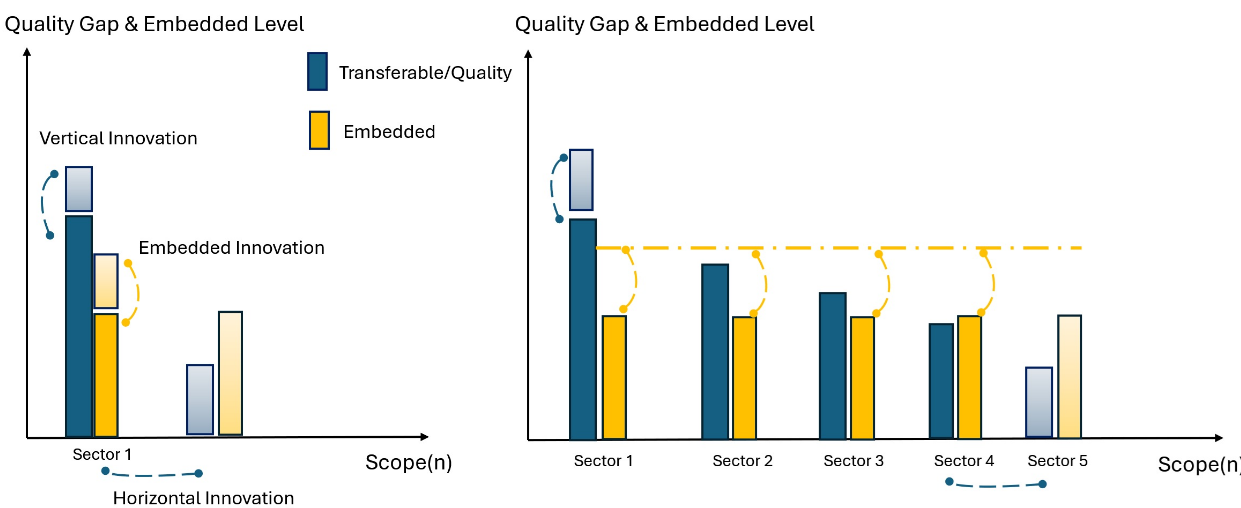

Figure 1. Firm Investment and Innovation Types

Superstar firms pursue three distinct innovation strategies. They can invest in transferable intangibles to either improve the quality of an existing product through vertical innovation or expand into a new sector through horizontal innovation. Additionally, they can invest in embedded intangibles to raise their brand value and organizational capital across all sectors in which they operate (see Figure 1). The first two strategies require sector-specific investment because product quality improvements embody knowl-

[9]Fringe firms are symmetric and of unit mass within each sector; sector-level aggregates equal individual quantities.

edge specific to each sector's technology. By contrast, embedded intangible investment is firm-specific, since an improvement in brand value or organizational capital simultaneously raises the embedded intangible stock of all sectors within the firm's portfolio, reflecting the economies of scope that embedded intangibles generate.

The variables I^{Emb}_{sjt}, I^{Ver}_{sjt}, and I^{Hor}_{sjt} denote the investment of superstar firm s in embedded intangibles, vertical innovation in an existing sector, and horizontal innovation into a new sector, respectively. Each unit of investment generates a successful flow rate of innovation, z^{Emb}_{sjt} for embedded intangibles, z^{Ver}_{sjt} for vertical, and z^{Hor}_{sjt} for horizontal innovation investment. Investments are subject to convex costs,

$$I^{\text{Ver}}_{sjt} = \gamma^{\text{Ver}} \left(z^{\text{Ver}}_{sjt}\right)^{\vartheta^{\text{Ver}}} Y_t, \qquad I^{\text{Hor}}_{jt} = \gamma^{\text{Hor}} \left(z^{\text{Hor}}_{sjt}\right)^{\vartheta^{\text{Hor}}} Y_t, \tag{8}$$

$$\text{and} \qquad I^{\text{Emb}}_{sjt} = \gamma^{\text{Emb}} \left(z^{\text{Emb}}_{sjt}\right)^{\vartheta^{\text{Emb}}} Y_t,$$

where the cost parameters γ^{Ver}, γ^{Hor}, and γ^{Emb} govern the scale of each investment cost function, ϑ^{Ver}, ϑ^{Hor}, and ϑ^{Emb} determine their curvature, and all costs scale with aggregate output Y_t.

Fringe firms invest only in transferable intangibles within their sector, targeting radical innovations that would allow them to displace the incumbent superstar. Their investment cost function takes the form

$$I_{fjt} = \gamma^f \left(z_{fjt}\right)^{\vartheta f} Y_t, \tag{9}$$

where γ^f is the cost scale parameter and ϑ^f determines the curvature of investment. Taken together, the total investment in transferable intangibles in sector j at time t combines vertical and horizontal investments by the incumbent superstar and investment by fringe firms,

$$I^{\text{T}}_{jt} = I^{\text{Ver}}_{sjt} + I^{\text{Hor}}_{sjt} + I_{fjt}. \tag{10}$$

A successful quality improvement, whether vertical within an existing sector or horizontal into a new sector, improves product quality by a factor of $\lambda > 1$, while a successful embedded intangible investment raises brand value and organizational capital by a factor of $\theta > 1$ across all active sectors simultaneously. Product quality and embedded intangibles therefore evolve as

$$q_{sjt} = \lambda^{m_{sjt}}\, q_{sj0}, \quad \text{and} \quad e_{st} = \theta^{k_{st}}\, e_{s0}, \tag{11}$$

where m_{sjt} denotes the cumulative number of product-quality innovations by firm s in sector j up to time t, and k_{st} denotes the cumulative number of embedded intangible innovations by firm s up to time t, with initial levels normalized to $q_{sj0} = 1$ and $e_{s0} = 1$. The quality gap between the superstar and the fringe in sector j at time t is therefore

$$\frac{q_{sjt}}{q_{fjt}} = \lambda^{m_{sjt} - m_{fjt}} = \lambda^{m_{jt}}, \tag{12}$$

where $m_{jt} \equiv m_{sjt} - m_{fjt}$ denotes the transferable intangible gap. Since the embedded intangible stock of fringe firms is normalized to one, the embedded intangible gap simplifies to the superstar's own stock,

$$\frac{e_{st}}{e_{ft}} = \frac{\theta^{k_{st}}}{1} = \theta^{k_{st}}. \tag{13}$$

Assumption (Managerial Reallocation Capability). *When a superstar firm — whether incumbent or challenger — faces competition, it can temporarily concentrate its entire managerial capacity on the contested sector. This reallocation is costless in the short run and determines the outcome of the price competition stage.*

Creative destruction arises from either a challenger superstar firm expanding its scope or a fringe firm pursuing radical innovation. When a superstar firm s successfully pursues horizontal innovation, it enters a randomly selected sector j' with incumbent superstar firm s', improving product quality by a factor λ such that $q_{sj't} = \lambda q_{s'j't}$, and displaces the incumbent s' if its effective unit cost — defined as the wage relative to total managerial capacity — falls below that of the incumbent,

$$\frac{w_t}{\Lambda\, q_{sj't}\, e_{st}^{\alpha}\, n_{st}^{1-\alpha}} < \frac{w_t}{\Lambda\, q_{s'j't}\, e_{s't}^{\alpha}\, n_{s't}^{1-\alpha}}.$$

The probability that a superstar firm s with state (k_{st}, n_{st}) successfully takes a new sector through horizontal innovation is

$$\mathbb{P}_{s>s'} \equiv \sum_{m_{s'}=1}^{\bar{m}} \sum_{k_{s'}=1}^{\bar{k}} \sum_{n_{s'}=1}^{\bar{n}} \mathbb{I}\{\lambda (\theta^{k_s})^{\alpha} n_{st}^{1-\alpha} > (\theta^{k_{s'}})^{\alpha} n_{s't}^{1-\alpha}\} \mu_t(m_{s'}, k_{s'}, n_{s'}),$$

such that the left-hand side reflecting the entrant's quality-adjusted managerial capacity after the λ improvement and the right-hand side reflecting the incumbent's. Here $\mu_t(m_{s'}, k_{s'}, n_{s'}) \in [0, 1]$ denotes the mass of superstar firms in state $(m_{s'}, k_{s'}, n_{s'})$ at time t. The state space is finite and discrete, given by $\mathcal{M} \times \mathcal{K} \times \mathcal{N}$, with $\mathcal{M} = \{1, \dots, \bar{m}\}$, $\mathcal{K} = \{1, \dots, \bar{k}\}$, $\mathcal{N} = \{1, \dots, \bar{n}\}$, and μ_t satisfies

$$\sum_{m_{s'}, k_{s'}, n_{s'}} \mu_t(m_{s'}, k_{s'}, n_{s'}) = 1. \tag{14}$$

Fringe firms cannot accumulate embedded intangibles and therefore cannot compete with the incumbent on managerial capacity. Their only path to becoming a superstar is through radical innovation, which requires jumping multiple steps on the quality ladder by a factor $\lambda^{\bar{m}} > \lambda$, raising the displacing firm's quality to $\lambda^{\bar{m}} q_{sjt}$, where q_{sjt} denotes the displaced incumbent's quality.[10][11] Since fringe firms are symmetric and invest identically within each sector, a fringe firm successfully displaces the superstar s in sector j if and only if

$$\mathbb{I}_{f>s} \equiv \mathbb{I}\{\lambda^{\bar{m}} > (\theta^{k_s})^{\alpha} n_s^{1-\alpha}\},$$

where the right-hand side depends solely on the incumbent's state (k_s, n_s).

The entry conditions through creative destruction show that the barrier is increasing in both the incumbent's embedded intangible stock θ^{k_s} and its scope n_s, regardless of whether the challenger is a fringe firm or a superstar. Moreover, the two embedded in-

[10] Fringe firms in the model correspond to small firms in the data. A vast literature documents that small and young firms grow faster than their larger counterparts (see Evans, 1987, Haltiwanger *et al.*, 2013).

[11] Following Akcigit *et al.* (2026), drastic innovation by fringe firms can alternatively be modeled using a probability distribution $\mathbb{F}(\cdot)$ over the innovation jumps, rather than a fixed step of size $\bar{m}$.

tangibles play asymmetric roles in the entry condition. Since only organizational capital enters total managerial capacity, eliminating it reduces entry barriers. By contrast, eliminating brand value raises entry barriers, as any increase in the embedded intangible stock then translates one-to-one into a stronger competitive advantage against challengers.

2.2. Equilibrium

The equilibrium has a static and a dynamic component. The analysis begins with the static equilibrium, determining prices and allocations for a given set of states. I then define the Markov Perfect Equilibrium for the dynamic game, outlining the value functions, optimal policy functions, and the evolution of the aggregate state distribution.

A household maximizes utility subject to the budget constraint, yielding the Euler equation

$$\frac{\dot{C}_t}{C_t} = r_t - \rho. \tag{15}$$

Along the balanced growth path, consumption and output grow at the same rate $g = r - \rho$, and the transversality condition holds. The final-good producer's demand for intermediate goods in sector j satisfies

$$p_{jt} \; = \; \frac{Y_t}{y_{jt}}. \tag{16}$$

This implies the demand functions for the superstar and fringe firms,

$$y_{sjt} = p_{jt}^{\frac{\varepsilon}{1-\varepsilon}} p_{sjt}^{\frac{1}{\varepsilon-1}} Y_t \big(\Xi(e_{st})\big)^{\frac{1}{1-\varepsilon}} \quad \text{and} \quad y_{fjt} = p_{jt}^{\frac{\varepsilon}{1-\varepsilon}} p_{fjt}^{\frac{1}{\varepsilon-1}} Y_t \big(1 - \Xi(e_{st})\big)^{\frac{1}{1-\varepsilon}}, \tag{17}$$

with p_{sjt} and p_{fjt} denoting the prices charged by the superstar and fringe firms, respectively. The corresponding ideal price index for sector j is

$$p_{jt} = \left(\big(\Xi(e_{st})\big)^{\frac{-1}{\varepsilon-1}} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} + \big(1 - \Xi(e_{st})\big)^{\frac{-1}{\varepsilon-1}} p_{fjt}^{\frac{\varepsilon}{\varepsilon-1}} \right)^{\frac{\varepsilon-1}{\varepsilon}}. \tag{18}$$

The Cobb-Douglas aggregator implies equal expenditure shares across sectors. The market share of superstar firm s in sector j at time t is

$$\frac{p_{sjt}y_{sjt}}{p_{jt}y_{jt}} = \frac{p_{sjt}y_{sjt}}{Y_t} = p_{jt}^{\frac{\varepsilon}{1-\varepsilon}} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} \, \Xi(e_{st})^{\frac{1}{1-\varepsilon}} \;\equiv\; \phi_{sjt}, \tag{19}$$

and since market shares sum to one, the fringe firms' share is $1 - \phi_{sjt}$. The equilibrium price of the superstar firm under Bertrand competition is[12]

$$p_{sjt} = \frac{1 - \varepsilon\,\phi_{sjt}}{(1 - \phi_{sjt})\,\varepsilon} \underbrace{\frac{w_t\, n_{st}^{\alpha}}{\Lambda\, q_{sjt}\, e_{st}^{\alpha}}}_{MC_{sjt}}, \tag{20}$$

where MC_{sjt} denotes the marginal cost of the superstar firm.[13] The equilibrium price increases with product quality and the embedded intangible stock and decreases with the number of sectors operated. The price ratio of fringe to superstar firms is

$$\frac{p_{fjt}}{p_{sjt}} = \frac{(1 - \phi_{sjt})\varepsilon}{1 - \varepsilon\phi_{sjt}} \cdot \lambda^{m_j} \left(\frac{(1 - \xi)e_{st}}{n_{st}} \right)^{\alpha}. \tag{21}$$

Substituting the ideal price index from equation (18) into the market share definition in equation (19), the superstar's market share can be expressed in terms of relative prices as

$$\phi_{sjt} \;=\; \frac{1}{1 + \left(\frac{1}{(\chi(e_{st}))^{\frac{1}{1-\varepsilon}}} \left(\frac{p_{fjt}}{p_{sjt}} \right)^{\frac{\varepsilon}{\varepsilon-1}} \right)}, \tag{22}$$

which depends on the quality gap m_j, the embedded intangible stock e_{st}, and the number of sectors operated n_{st}.

The operational profit and markup of the superstar firm are proportional to its market share,

$$\pi_{sjt} = \frac{(1 - \varepsilon)\,\phi_{sjt}}{1 - \varepsilon\phi_{sjt}}\, Y_t \qquad \text{and} \qquad \sigma_{sjt} = \frac{1 - \varepsilon\phi_{sjt}}{(1 - \phi_{sjt})\,\varepsilon}. \tag{23}$$

Both profit and markup increase with market share and therefore decrease with firm

[12] See Appendix B.2 for the full derivations. For the à la Cournot competition, see Appendix B.3.

[13] Although fringe firms lack independent pricing power, their presence creates a competitive constraint that limits the superstar firm's pricing power and prevents it from extracting monopoly rents.

scope, reflecting the diminishing managerial productivity that arises from the span-of-control friction. The optimal labor inputs for superstar and fringe firms are

$$l_{sjt} = \frac{\phi_{sjt}}{\sigma_{sjt}} \, \omega_t^{-1} \quad \text{and} \quad l_{fjt} = (1 - \phi_{sjt}) \, \omega_t^{-1}, \tag{24}$$

where $\omega_t = w_t / Y_t$ denotes the wage share of the economy. The static equilibrium provides only an implicit solution. Nonetheless, the model yields tractable dynamics because the equilibrium outcomes for both superstar and fringe firms depend solely on their market share, which is determined by the quality gap, the embedded intangible stock, and the number of sectors in which the superstar operates.

The value function of superstar firm s depends on the quality gap vector $\mathbf{m} = \{m_j\}_{j=1}^n$, the embedded intangible level k, and the number of sectors operated n. Although the firm operates across multiple sectors, vertical and horizontal innovation choices are made at the sector level because each sector's quality gap evolves independently. By contrast, embedded intangible investment raises k simultaneously across all active sectors, so the firm internalizes its joint effect on every sector when setting z^{Emb}.

The superstar firm chooses innovation flow rates[14] z_j^{Ver}, z_j^{Hor}, and z_j^{Emb} to maximize

$$\begin{aligned} r_t V_t(\mathbf{m}, k, n) - \dot{V}_t(\mathbf{m}, k, n) = \max_{\{z_{jt}^{\text{Ver}}, z_{jt}^{\text{Hor}}, z_{jt}^{\text{Emb}}\}_{j=1}^n} \sum_{j=1}^{n} \Bigg(& \frac{(1-\varepsilon)\,\phi_t(m_j,k,n)}{1-\varepsilon\,\phi_t(m_j,k,n)} \, Y_t \\ & + \underbrace{\mathbb{P}_{s>s',t} \, z_{jt}^{\text{Hor}} \Big[V_{jt}\big((m_j,1),k,n+1\big) - V_{jt}(m_j,k,n) \Big]}_{\text{Horizontal Innovation}} + \underbrace{z_{jt}^{\text{Ver}} \Big[V_{jt}(m_j+1,k,n) - V_{jt}(m_j,k,n) \Big]}_{\text{Vertical Innovation}} \\ & + \underbrace{z_{jt}^{\text{Emb}} \Big[V_{jt}(\mathbf{m},k+1,n) - V_{jt}(\mathbf{m},k,n) \Big]}_{\text{Embedded Innovation}} + \underbrace{Z_{c>s,t}^{\text{Hor}} \Big[- V_{jt}(m_j,k,n) \Big]}_{\text{Challenger Superstar Entry}} + \underbrace{\mathbb{I}_{f>s,t} \, Z_{jt}^{f} \Big[- V_{jt}(m_j,k,n) \Big]}_{\text{Fringe Entry}} \\ & - \gamma^{\text{Ver}} \left(z_{jt}^{\text{Ver}}\right)^{\vartheta^{\text{Ver}}} Y_t - \gamma^{\text{Emb}} \left(z_{jt}^{\text{Emb}}\right)^{\vartheta^{\text{Emb}}} Y_t - \gamma^{\text{Hor}} \left(z_{jt}^{\text{Hor}}\right)^{\vartheta^{\text{Hor}}} Y_t \Bigg). \end{aligned} \tag{25}$$

The left-hand side represents the flow return on the firm's value. The first term on the right-hand side is the operational profit of the superstar firm in sector j. The second and

[14] To simplify notation, the subscript s is omitted whenever it is clear from context.

third terms capture the value gains from horizontal and vertical innovation, respectively. A successful horizontal innovation expands the firm's scope from n to $n+1$, while a successful vertical innovation advances the quality gap in sector j by one rung. The fourth term captures the gain from a successful embedded intangible investment, which raises the embedded gap k by one rung across all active sectors simultaneously. The fifth and sixth terms capture the value loss upon displacement by a challenger superstar or a fringe firm. The final three terms are the convex investment costs defined in equation (8). The terms $Z^{\text{Hor}}_{c>s,t}$ and Z^{f}_{jt} denote the aggregate innovation rates of challenger superstar firms and fringe firms, respectively. The aggregate horizontal innovation rate of challenger superstars that successfully displace the incumbent is

$$Z^{\text{Hor}}_{c>s,t} = \sum_{m_c, k_c, n_c} \mathbb{I}\left\{\lambda\,(\theta^{k_c})^{\alpha}\, n_{ct}^{1-\alpha} > (\theta^{k_s})^{\alpha}\, n_{st}^{1-\alpha}\right\} z^{\text{Hor}}_{t}(m_c, k_c, n_c)\,\mu_t(m_c, k_c, n_c),$$

and the aggregate fringe firm innovation rate is $Z^{f}_{jt} = \int_0^1 z_{fjt}\, df$.

Along the balanced growth path, aggregate output Y, consumption C, and the value function $V(\mathbf{m}, k, n)$ all grow at the constant rate g. Dividing the value function by aggregate output and defining $v(\mathbf{m}, k, n) \equiv V(\mathbf{m}, k, n)/Y$ yields the normalized superstar value function, which is given

$$\begin{aligned}
\rho\, v(\mathbf{m}, k, n) = \max_{\{z_j^{\text{Ver}}, z_j^{\text{Hor}}, z_j^{\text{Emb}}\}_{j=1}^{n}} \sum_{j=1}^{n} \Bigg(& \frac{(1-\varepsilon)\,\phi(m_j, k, n)}{1-\varepsilon\,\phi(m_j, k, n)} \\
& + \underbrace{\mathbb{P}_{s>s'}\, z_j^{\text{Hor}}\Big[v\big((m_j, 1), k, n+1\big) - v(m_j, k, n)\Big]}_{\text{Horizontal Innovation}} + \underbrace{z_j^{\text{Ver}}\Big[v(m_j+1, k, n) - v(m_j, k, n)\Big]}_{\text{Vertical Innovation}} \\
& + \underbrace{z_j^{\text{Emb}}\Big[v(\mathbf{m}, k+1, n) - v(\mathbf{m}, k, n)\Big]}_{\text{Embedded Innovation}} + \underbrace{Z^{\text{Hor}}_{c>s}\Big[-v(m_j, k, n)\Big]}_{\text{Challenger Superstar Entry}} + \underbrace{\mathbb{I}_{f>s}\, Z_j^{f}\Big[-v(m_j, k, n)\Big]}_{\text{Fringe Entry}} \\
& - \gamma^{\text{Ver}}\left(z_j^{\text{Ver}}\right)^{\vartheta^{\text{Ver}}} - \gamma^{\text{Emb}}\left(z_j^{\text{Emb}}\right)^{\vartheta^{\text{Emb}}} - \gamma^{\text{Hor}}\left(z_j^{\text{Hor}}\right)^{\vartheta^{\text{Hor}}} \Bigg).
\end{aligned} \tag{26}$$

Since fringe firms produce a homogeneous good, they earn zero profit in equilibrium and derive value solely from the possibility of displacing the incumbent through radical innovation. The fringe firm chooses innovation intensity z_f to maximize this value, trading off investment costs against the gain from becoming the new superstar. The normalized fringe firm value function $v_f(m,k,n) \equiv V_{ft}(m,k,n)/Y$ in the balance growth path satisfies

$$\rho\, v_f(m,k,n) = \max_{z_f} \Bigg\{ \underbrace{\mathbb{I}_{f>s}\, z_f \Big[v(\bar{m},1,1) - v_f(m,k,n) \Big]}_{\text{Radical Innovation}} - \gamma^f (z_f)^{\vartheta^f}$$
$$+ \underbrace{\mathbb{P}_{s>s'}\, z_j^{\text{Hor}} \Big[v_f\big((m,1),k,n+1\big) - v_f(m,k,n) \Big]}_{\text{Incumbent Horizontal Innovation}} + \underbrace{z_j^{\text{Ver}} \Big[v_f(m+1,k,n) - v_f(m,k,n) \Big]}_{\text{Incumbent Vertical Innovation}}$$
$$+ \underbrace{z_j^{\text{Emb}} \Big[v_f(m,k+1,n) - v_f(m,k,n) \Big]}_{\text{Incumbent Embedded Innovation}} + \underbrace{\mathbb{E}_{\mu}\left[\mathbb{I}_{c>s} \cdot z^{\text{Hor}} \cdot \Delta v_f\right]}_{\text{Challenger Superstar Entry}} + \underbrace{\mathbb{I}_{f>s}\, Z_j^f \Big[v_f(\bar{m},1,1) - v_f(m,k,n) \Big]}_{\text{Fringe Entry}} \Bigg\}, \tag{27}$$

where the challenger superstar entry term is

$$\mathbb{E}_{\mu}\left[\mathbb{I}_{c>s} \cdot z^{\text{Hor}} \cdot \Delta v_f\right] \equiv \sum_{m_c,k_c,n_c} \mathbb{I}\left\{ \lambda(\theta^{k_c})^{\alpha} n_c^{1-\alpha} > (\theta^{k_s})^{\alpha} n_s^{1-\alpha} \right\} z^{\text{Hor}}(m_c,k_c,n_c)\, \mu(m_c,k_c,n_c)\, \Delta v_f,$$

with $\Delta v_f \equiv v_f(1,k_s,n_s) - v_f(m,k,n)$ denoting the value change for the fringe firm upon facing a new incumbent. The first term captures the value gain from radical innovation. The indicator $\mathbb{I}_{f>s}$ equals one when the fringe firm's innovation is sufficient to overcome the incumbent's total managerial capacity, upon which it enters as a new superstar with state $(\bar{m},1,1)$. The remaining terms account for changes in the competitive environment arising from the incumbent's own innovations, challenger superstar entry, and successful radical innovation by another fringe firm, each of which alters the state the fringe firm faces.

The first-order conditions of the superstar and fringe value functions in equations (26) and (27) yield the following optimal innovation rates along the balanced growth path,

$$z_j^{\text{Hor}} = \left(\frac{\mathbb{P}_{s>s'} \left[v_j\big((m_j,1),k,n+1\big) - v_j(m_j,k,n) \right]}{\gamma^{\text{Hor}} \cdot \vartheta^{\text{Hor}}} \right)^{\frac{1}{\vartheta^{\text{Hor}}-1}} \tag{28}$$

$$z_j^{\text{Ver}} = \left(\frac{v_j(m_j+1,k,n) - v_j(m_j,k,n)}{\gamma^{\text{Ver}} \cdot \vartheta^{\text{Ver}}} \right)^{\frac{1}{\vartheta^{\text{Ver}}-1}} \tag{29}$$

$$z_j^{\text{Emb}} = \left(\frac{v(\mathbf{m},k+1,n) - v(\mathbf{m},k,n)}{\gamma^{\text{Emb}} \cdot \vartheta^{\text{Emb}}} \right)^{\frac{1}{\vartheta^{\text{Emb}}-1}} \tag{30}$$

$$z_f = \left(\frac{\mathbb{I}_{f>s} \left[v(\bar{m},1,1) - v_f(m,k,n) \right]}{\gamma^{f} \cdot \vartheta^{f}} \right)^{\frac{1}{\vartheta^{f}-1}} \tag{31}$$

Each optimal innovation rate is increasing in the value gain from the corresponding innovation and decreasing in its cost scale and curvature parameters.

The law of motion for the distribution $\mu_t(m,k,n)$ is given by

$$\begin{aligned} \dot{\mu}_t(m,k,n) = {} & z_t^{\text{Ver}}(m-1,k,n)\,\mu_t(m-1,k,n) + z_t^{\text{Emb}}(m,k-1,n)\,\mu_t(m,k-1,n) \\ & + \mathbb{P}_{s>s'}\, z_t^{\text{Hor}}(m,k,n-1)\,\mu_t(m,k,n-1) - z_t^{\text{Ver}}(m,k,n)\,\mu_t(m,k,n) \\ & - z_t^{\text{Emb}}(m,k,n)\,\mu_t(m,k,n) - \mathbb{P}_{s>s'}\, z_t^{\text{Hor}}(m,k,n)\,\mu_t(m,k,n) \\ & - \left(\mathbb{I}_{f>s}\, Z_t^{f}(m,k,n) + Z_{c>s,t}^{\text{Hor}} \right) \mu_t(m,k,n). \end{aligned} \tag{32}$$

Inflows arise when a firm in an adjacent predecessor state successfully innovates: vertical innovation moves a sector from $(m-1,k,n)$ to (m,k,n), embedded intangible investment moves it from $(m,k-1,n)$ to (m,k,n), and horizontal innovation by a superstar with $n-1$ sectors raises its scope to n.[15] Outflows arise when the incumbent itself innovates and transitions to a higher state, or when it is displaced by either a fringe firm or a challenger superstar. At the boundaries of the state space, states that reach the upper bounds $(\bar{m},\bar{k},\bar{n})$ are absorbing in the corresponding dimension. There are two special inflows: a successful fringe radical innovation creates a new superstar with state $(\bar{m},1,1)$, while a successful superstar horizontal innovation enters a new sector with quality gap $m=1$, generating an inflow at $(1,k,n)$.

[15] Only single-event transitions govern the law of motion, since simultaneous innovations across distinct dimensions occur with probability $o(\Delta t)$. Because $I^{\text{Ver}}, I^{\text{Hor}}, I^{\text{Emb}}$ are independent Poisson processes with arrival rates $z^{\text{Ver}}, z^{\text{Hor}}, z^{\text{Emb}}$, $\Pr(\text{both } z^i \text{ and } z^j \text{ arrive in } [t,t+\Delta t]) = z^i z^j (\Delta t)^2 = o(\Delta t)$.

The labor market clears according to

$$1 = \int_0^1 \left(l_{sjt} + l_{fjt}\right) dj. \tag{33}$$

Using equations (24) and (33), the normalized wage satisfies

$$\omega_t = \sum_{m=1}^{\bar{m}} \sum_{k=1}^{\bar{k}} \sum_{n=1}^{\bar{n}} \left(\frac{\phi_t(m,k,n)}{\sigma_t(m,k,n)} + 1 - \phi_t(m,k,n) \right) \mu_t(m,k,n). \tag{34}$$

Combining the production functions in equations (3), (6), and (7) with the labor demand equations (24) yields aggregate output

$$Y_t = Q_t\, \omega_t^{-1} \exp\left(\int_0^1 \ln \left[\Xi\left(e_{st}\right) \left(\frac{((1-\xi)e_{st})^{\alpha}}{\eta\, n_{st}^{\alpha}} \frac{\phi_{sjt}}{\sigma_{sjt}} \right)^{\varepsilon} + (1-\Xi(e_{st})) \left(\lambda^{-m_{jt}} (1-\phi_{sjt}) \right)^{\varepsilon} \right]^{\frac{1}{\varepsilon}} dj \right), \tag{35}$$

where $Q_t = \exp\Big(\int_0^1 \ln q_{sjt}\, dj \Big)$ denotes the aggregate quality. Along the balanced growth path, the economy grows at the rate[16]

$$g \;=\; \ln \lambda \sum_{m=1}^{\bar{m}} \sum_{k=1}^{\bar{k}} \sum_{n=1}^{\bar{n}} \left(z^{\mathrm{Ver}}(m,k,n) + \mathbb{P}_{s>s'}\, z^{\mathrm{Hor}}(m,k,n) + Z^{f}(m,k,n) \right) \mu(m,k,n). \tag{36}$$

Finally, the resource constraint requires that aggregate output is allocated between consumption and investment across all sectors,

$$Y_t = C_t + \int_0^1 \left(I_{jt}^{\mathrm{Ver}} + I_{jt}^{\mathrm{Hor}} + I_{jt}^{\mathrm{Emb}}\right) dj + \int_0^1 I_{jt}^{f}\, dj, \quad \text{with} \quad I_{jt}^{f} = \int_{F_j} I_{ijt}\, di. \tag{37}$$

Definition (Markov Perfect Equilibrium). *A Markov Perfect Equilibrium consists of an allocation $\{C_t, Y_t, y_{sjt}, y_{fjt}\}$, prices $\{r_t, w_t, p_{sjt}, p_{fjt}\}$, innovation policies $\{z^{Ver}, z^{Hor}, z^{Emb}, z_f\}$, labor allocations $\{l_{sjt}, l_{fjt}\}$, and a distribution $\mu_t(m,k,n)$ such that: (i) the final good producer maximizes profit given prices; (ii) the superstar firm maximizes its value function given state $(\boldsymbol{m}, k, n)$, choosing innovation rates as in equations (28)–(30); (iii) the fringe firm chooses its*

[16] See Appendix B.4 for details.

optimal innovation rate as in equation (31)*; (iv) the labor market clears; (v) the distribution* $\mu_t(m, k, n)$ *satisfies equation* (14) *and evolves according to equation* (32)*; and (vi) aggregate consumption and output grow at the common rate* g *given by equation* (36)*, with the resource constraint* (37) *satisfied.*

3. Dataset and Empirical Facts

This section serves two purposes. First, it describes the dataset and measurement choices used throughout the paper. Second, it presents local projections that document how single- and multi-sector firms differ in their response to innovation.

3.1. Data Description

Compustat Fundamentals provides comprehensive firm-level financial information for publicly listed companies in North America,[17] with longitudinal coverage of balance sheet items, income statement components, and cash flow data. Measuring firm scope additionally requires information on the industries in which a firm operates. I use the Compustat Historical Segment file, which compiles firms' mandated segment-level disclosures from 1976 to the present.[18] Compustat assigns a unique segment identifier (`sid`) to each segment a firm reports, and I define firm scope as the number of distinct segments reported in a given year. Under SFAS 131's management approach, segment granularity is determined internally by the firm and varies in practice from narrow four-digit SIC distinctions to broader product or divisional groupings, so the scope measure reflects each

[17] Compustat North America includes foreign firms cross-listed in the U.S. via American Depositary Receipts (ADRs), such as Toyota and Unilever. These firms are subject to SEC regulation, comply with U.S. reporting requirements, compete in U.S. product markets, and obtain patents from the U.S. Patent and Trademark Office, making them economically relevant for this study. Excluding them reduces the sample size by approximately 15% without altering the main conclusions, so they are retained throughout. Table A3 compares summary statistics with and without ADRs.

[18] U.S. public firms are required to disclose segment-level information in their 10-K filings. From 1976 to 1997, these disclosures were governed by SFAS No. 14, under which segments were defined using an *industry approach*. From fiscal year 1998 onward, segments are reported under SFAS No. 131. Firms without segment records in a given year are excluded from the sample.

firm's own delineation of its lines of business rather than a fixed industry partition.[19] I merge this scope measure with Compustat Fundamentals to construct the firm-level measures of markups, productivity, and intangible composition used below.

To capture the competitive pressure firms face, I use the product-market fluidity metric of Hoberg *et al.* (2014), which measures how rapidly rivals in similar product markets change their offerings. The metric is constructed using natural language processing on product descriptions from firms' annual 10-K filings, tracking year-over-year changes in how companies describe their business activities. A high fluidity score indicates that rivals are rapidly reconfiguring their market positions, intensifying the competitive pressure the firm faces.

To track how firm outcomes evolve around innovation shock, I use the patent-value dataset of Kogan *et al.* (2017), which provides patent identifiers, filing and grant dates, firm identifiers, forward citations, and a market-based estimate of patent value. The patent-value measure is constructed from stock-market reactions within a narrow event window around patent grant dates, capturing the surprise component of the market's response to each grant. The short-window design isolates the incremental information content of the grant itself, abstracting from anticipation effects and concurrent firm-level news.

3.2. Measurement

I estimate firm-level total factor productivity using the proxy-variable approach of Olley and Pakes (1996), which addresses the simultaneity between input choices and unobserved productivity by exploiting the monotonicity of investment in productivity conditional on capital. Under this assumption, the firm's investment policy function is invertible in productivity, so the productivity shock can be expressed as a function of observ-

[19] The Historical Segment dataset reports several segment classifications at the firm-year level, including business, operating, and geographic segments. Beginning in 2015, the WRDS Historical Segment files additionally provide product–service (`PD_SRVC`) information, which directly identifies the products and services associated with each segment. To construct a consistent measure of firm scope, I therefore use business segments prior to 2015 and `PD_SRVC`-based segment definitions thereafter. Table A1 provides an illustrative comparison of the resulting segment classifications.

able investment and capital and substituted into the production function.[20] Firm-level markups follow De Loecker *et al.* (2020): $\sigma_{it} \equiv \beta^V_{it}/s^V_{it}$, where β^V_{it} is the output elasticity of the variable input (intermediates) recovered from the production function estimation and s^V_{it} is the variable input's expenditure share in revenue.

Following Peters and Taylor (2017), I measure transferable intangible investment as total R&D expenditure and embedded intangible investment as 30% of Selling, General, and Administrative (SG&A) expenses net of R&D. SG&A encompasses a broad range of expenditures including advertising, employee compensation, and general operational costs. Once R&D is netted out, the remaining SG&A primarily reflects advertising and employee-related costs, most of which are routine operating expenses consumed in the current period. A portion, however, accumulates within the firm as lasting intangible capital: advertising builds brand recognition, and employee compensation develops organizational knowledge and managerial talent. The 30% fraction captures this accumulating component, with the remainder treated as routine operating costs that generate no lasting intangible value.[21]

3.3. Empirical Facts

The model implies that single-sector firms, whose narrow scope concentrates innovation incentives within a single market, should reallocate more aggressively toward transferable intangibles, face stronger competitive entry, and capture larger productivity and markup gains than multi-sector firms, whose scope and accumulated embedded intangibles attenuate the response on each margin. To assess whether these predicted differences are present in the data, I estimate local projections following Jordà (2005), using the patent-value measure of Kogan *et al.* (2017) as the innovation shock. The patent-value

[20] For methodological details, see Appendix A.2.1. Table A4 reports elasticity estimates under the alternative Ackerberg *et al.* (2015) and Gandhi *et al.* (2020) estimators.

[21] The two components of embedded intangibles — brand value and organizational capital — could in principle be identified separately, since advertising expenditure is reported as a distinct line item (XAD) in Compustat. However, XAD is sparsely reported, and conditioning on non-missing values would disproportionately retain firms with large advertising budgets, introducing a systematic selection toward advertising-intensive firms.

measure isolates the unexpected component of grant-related news, and grant timing is largely predetermined relative to the firm's contemporaneous strategic decisions.[22] The local projection specification takes the form

$$\Delta_h Y_{ijt} = \alpha_h + \beta_{\text{Multi}} V_{it} D_{it} + \beta_{\text{Single}} V_{it}(1 - D_{it}) + \mathbf{\Gamma}_h^\top \mathbf{X}_{i,t-1} + \delta_h Y_{i,t-1} + \theta_j + \lambda_t + u_{ijt}, \quad (38)$$

where $\Delta_h Y_{ijt} \equiv Y_{i,t+h} - Y_{it}$ is the h-period change in outcome Y for firm i in industry j, $V_{it} = \sum_{p \in \mathcal{P}_{it}} \text{PatentValue}_{p,t}$ is the firm-year sum of innovation values, and $D_{it} = \mathbf{1}\{i \text{ is multi-sector at } t\}$ indicates multi-sector status. The vector $\mathbf{X}_{i,t-1}$ collects lagged controls, $Y_{i,t-1}$ absorbs pre-shock outcome levels, and θ_j and λ_t denote industry and year fixed effects. The coefficients β_{Single} and β_{Multi} trace the impulse responses of single- and multi-sector firms to an innovation shock. Scope status is defined using each firm's pre-shock classification and held fixed throughout the estimation window.

Single- and multi-sector firms differ in technology, industry composition, and prior intangibles, all of which jointly shape both the distribution of patent values and the response to a given shock. The impulse responses therefore document how outcomes evolve differently across the scope distribution rather than identifying the causal effect of scope.

Innovation shocks elicit qualitatively similar response patterns across firm types, but the magnitudes differ sharply on every margin. Single-sector firms reallocate investment toward transferable intangibles immediately after a shock, and the response widens monotonically over the subsequent three years (Figure 2A). Multi-sector firms exhibit a substantially weaker reallocation, consistent with the model's prediction that operating across multiple sectors dilutes the incentive to concentrate innovative investment in any single domain.

Competitive threat follows the same pattern. Single-sector innovators face significant rival entry following an innovation shock, while multi-sector firms exhibit no measurable change in competitive pressure (Figure 2B). The contrast aligns with the model's mecha-

[22] Segment changes are infrequent in the data, suggesting that the innovation shocks identified here predominantly reflect vertical improvements within existing sectors rather than scope-expanding horizontal innovations.

nism: scope and accumulated embedded intangibles raise entry barriers around multi-sector firms, insulating them from the competitive spillovers that single-sector innovators face.

(A) Investment Ratio **(B)** Competitive Pressure

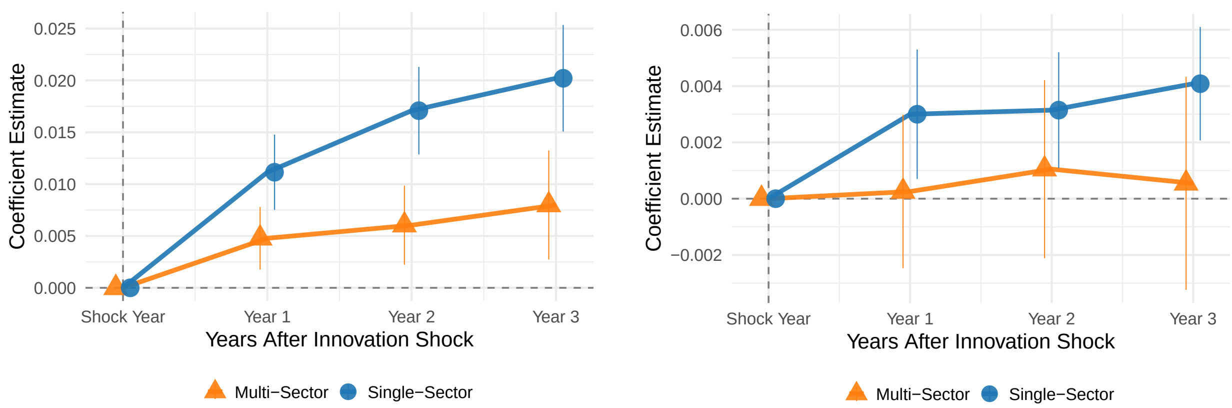

Figure 2. Investment Ratio and Competitive Pressure Responses to Innovation Shocks

Note: The innovation shock enters as $\log(1 + V_{it})$ to retain observations with zero patent grants. Outcomes are defined as the h-year change in logs, $\Delta_h \log(Y)$, and controls also enter in logarithms. The sample excludes utilities and finance, as well as firms with missing or non-positive R&D, SG&A, employment, or sales. All variables cover the period 1990–2019. Vertical bars denote 95% confidence intervals based on standard errors clustered at the industry-year level.

The contrast extends to productivity and markups, the margins that most directly capture appropriability. Single-sector firms display a strong and monotonically increasing productivity response, with the coefficient rising steadily from the shock year through year three (Figure 3A). Multi-sector firms, by contrast, exhibit a response that remains flat and statistically indistinguishable from zero throughout the horizon, and the gap between the two groups widens persistently.

A parallel but more moderate pattern emerges for markups (Figure 3B). Single-sector firms command significantly higher price-cost margins following an innovation shock, with the markup response growing monotonically and precisely estimated at every horizon. Multi-sector firms also exhibit a positive and increasing markup response, but the magnitude is substantially smaller, and the gap between the two groups widens persistently from year one onward. Single-sector firms therefore translate innovation shocks into lasting gains in productive efficiency and market power, while multi-sector firms display materially smaller responses on both margins. The pattern is consistent with Autor *et al.* (2020), in which higher-productivity firms command higher markups, and extends it by showing that the productivity-markup co-movement is concentrated in single-sector firms.[23]

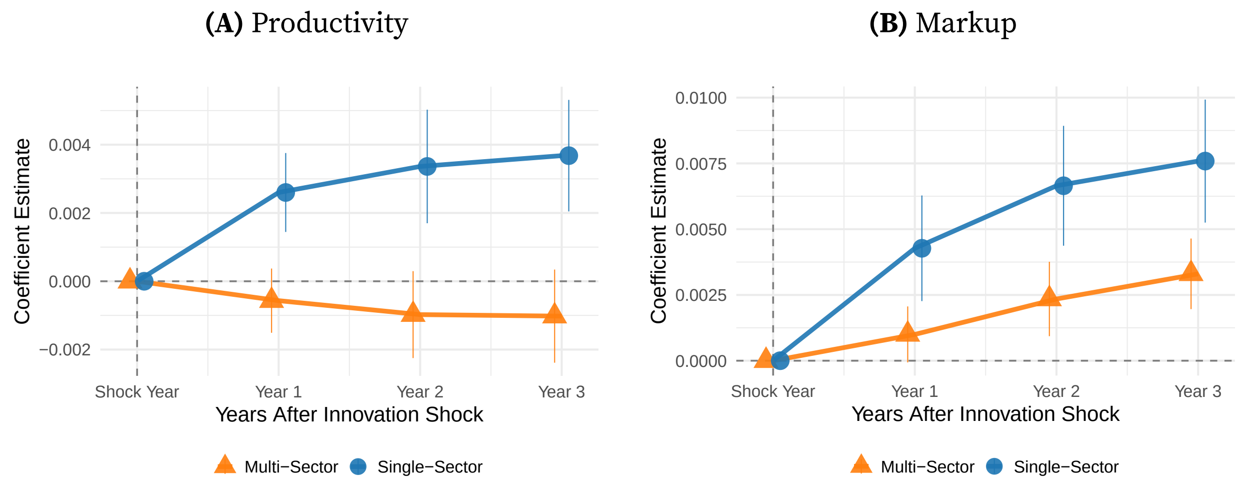

Figure 3. Productivity and Markup Responses to Innovation Shocks

Note: The innovation shock enters as $\log(1 + V_{it})$ to retain observations with zero patent grants. Outcomes are defined as the h-year change in logs, $\Delta_h \log(Y)$, and controls also enter in logarithms. The sample excludes utilities and finance, as well as firms with missing or non-positive R&D, SG&A, employment, or sales. All variables cover the period 1990–2019. Vertical bars denote 95% confidence intervals based on standard errors clustered at the industry-year level.

Three empirical facts[24] confirm that the differential responses predicted by the model are present in the data. Section 4 uses cross-sectional moments to discipline the structural model and provide a quantitative test of the mechanism.

Fact 1. In response to an innovation shock, single-sector firms reallocate more sharply and persistently toward transferable intangibles than multi-sector firms.

Fact 2. An innovation shock raises competitive pressure on single-sector firms but produces no significant response in multi-sector firms.

Fact 3. The productivity and markup responses to an innovation shock are concentrated in single-sector firms, with multi-sector firms displaying significantly weaker responses

[23] Appendix Figure A1 replicates the analysis using productivity and markup estimates based on Ackerberg *et al.* (2015) and Gandhi *et al.* (2020).

[24] Appendix Table A2 also shows a two-way fixed-effects specification.

on both margins.

4. Quantitative Analysis

4.1. Calibration and Identification

The model is solved on the balanced growth path (BGP) and calibrated to match moments that vary systematically across the firm-scope distribution. Because the state space is three-dimensional, each scope-conditional moment is constructed by integrating the relevant firm-level object over the joint distribution of (m,k) at fixed scope n, with the scope distribution given by

$$\mu(n) \;=\; \sum_m \sum_k \mu(m,k,n). \tag{39}$$

The competitive threat at scope n is the probability that an incumbent loses a given sector, summing displacement by a rival superstar and by a fringe entrant:

$$CT(n) \;=\; \mathbb{E}_n[\mathbb{P}_{c>s}(m,k,n) \;+\; \mathbb{I}_{f>s}(k,n)]\,. \tag{40}$$

The transferable investment share and the average markup at scope n are

$$I(n) \;=\; \mathbb{E}_n\left[\frac{I^T(m,k,n)}{I^{\text{Emb}}(m,k,n)+I^T(m,k,n)}\right], \quad \sigma(n) = \mathbb{E}_n\left[\frac{1-\varepsilon\,\phi(m,k,n)}{(1-\phi(m,k,n))\,\varepsilon}\right], \tag{41}$$

where $\phi(m,k,n)$ is the fringe expenditure share derived in Section 2.[25] The productivity growth rate aggregates the contributions of vertical, horizontal, and radical innovation,

$$g(n) \;=\; \mathbb{E}_n\Big[\ln\lambda\cdot\Big(z^{\text{Ver}}(m,k,n)+\mathbb{P}_{s>s'}(m,k,n)\,z^{\text{Hor}}(m,k,n)+\mathbb{I}_{f>s}(k,n)\,Z^f(m,k,n)\Big)\Big]\,, \tag{42}$$

with the corresponding gross productivity factor $Q(n)=e^{g(n)}$.[26]

[25] The ratio I^T/I^{Emb} is undefined at the boundary $k=\bar{k}$, where $I^{\text{Emb}}=0$. The share $I^T/(I^{\text{Emb}}+I^T)$ remains well-defined throughout the state space.

[26] Productivity satisfies $\dot{Q}(n,t)/Q(n,t)=g(n)$, integrating to $Q(n,t)=Q(n,t_0)\,e^{g(n)(t-t_0)}$ up to a constant absorbed into the initial condition. The normalization affects only levels and has no bearing on equilibrium

The estimation procedure uses 36 moments, comprising 18 targeted and 18 untargeted moments computed at six scope levels, to discipline 16 model parameters. Four parameters are set externally. The time discount rate is fixed at $\rho = 0.05$, and the curvature parameters governing the cost of transferable innovation are set to $\vartheta^{\text{Ver}} = \vartheta^{\text{Hor}} = \vartheta^{f} = 2.0$, following Akcigit and Kerr (2018). The remaining twelve parameters,

$$\left(\varepsilon,\, \lambda,\, \theta,\, \alpha,\, \eta,\, \beta,\, \xi,\, \gamma^{\text{Ver}},\, \gamma^{\text{Hor}},\, \gamma^{f},\, \gamma^{\text{Emb}},\, \vartheta^{\text{Emb}}\right),$$

are estimated jointly by minimizing the distance between model and empirical moments across scope levels.[27]

Table 1. Parameter Values

External Calibration			Internal Calibration		
Parameter	**Description**	**Value**	**Parameter**	**Description**	**Value**
ρ	Discount rate	0.050	ε	CES parameter	0.881
ϑ^{Ver}	Vertical invest. curvature	2.000	λ	Quality step size	1.131
ϑ^{Hor}	Horizontal invest. curvature	2.000	θ	Embedded step size	1.143
ϑ^{f}	Fringe invest. curvature	2.000	α	Managerial curvature	0.380
			η	Managerial scale	0.994
			β	Brand curvature	0.031
			ξ	Brand share	0.247
			γ^{Ver}	Vertical cost scale	9.910
			γ^{Hor}	Horizontal cost scale	4.654
			γ^{f}	Fringe cost scale	249.191
			γ^{Emb}	Embedded cost scale	0.762
			ϑ^{Emb}	Embedded curvature	2.032

Note: The upper limit for the number of production lines $\bar{n}$ is set to 6, and the upper bounds for $\bar{m}$ and $\bar{k}$ are set to 7 and 9, respectively.

Although estimation is joint, each parameter is most directly disciplined by a distinct feature of the moment vector. The CES elasticity ε pins down the level of price-cost margins and is identified from the cross-sectional mean of $\sigma(n)$. The managerial-productivity parameters α and η shape how markups decline with scope through the span-of-control friction on horizontal expansion, so they are jointly identified from the slope of $\sigma(n)$ across scope levels. The quality step size λ enters $g(n)$ linearly conditional on the equi-

allocations or identification.

[27] Appendix C.2 describes the solution and estimation algorithm.

librium innovation rates and is therefore identified from the level of the growth rate. The cost-scale parameters $\gamma^{\text{Ver}}, \gamma^{\text{Hor}}, \gamma^{f}$, and γ^{Emb}, together with the embedded-investment curvature ϑ^{Emb} and the step size θ, govern the marginal cost of each innovation strategy and are jointly identified from the scope distribution $\mu(n)$ and the transferable investment share $I(n)$, which together pin down the equilibrium allocation of firms across the state space. The composition parameter ξ determines the share of the embedded intangible stock allocated to organizational capital. Because this share governs the incumbent's ability to deter entry, ξ is identified from the competitive threat $CT(n)$. The remaining parameter β shapes the curvature of brand value, entering the investment share and the markup without affecting competitive threat. It is therefore identified from the nonlinearity of $I(n)$ and $\sigma(n)$ in the embedded intangible stock across scope levels.[28]

A potential concern in calibrating a model with multiple intangible margins is that complementarities between investment types could admit multiple equilibria. Transferable and embedded intangibles are jointly supermodular: a higher embedded stock raises the marginal return to vertical quality improvement, and vice versa.[29] This complementarity distinguishes the present setting from quality-ladder frameworks in the tradition of Akcigit and Ates (2023), where decreasing returns to innovation in each margin generate stable dynamics that endogenously bound the equilibrium state distribution. Here, each intangible margin exhibits decreasing returns when the other is held fixed, but the positive cross-partial between m and k produces cross-margin amplification: investment in transferable intangibles raises the marginal return to embedded accumulation, and investment in embedded intangibles raises the marginal return to vertical improvement. Absent explicit bounds on the state, this complementarity could in principle generate either unbounded investment dynamics or multiple BGP equilibria. [30]

[28] Appendix C1 reports moment elasticities with respect to each parameter.

[29] Figure C5 plots the joint behavior of the value function across all pairwise combinations of the state space, $V_s(m,k)$, $V_s(m,n)$, and $V_s(k,n)$.

[30] The flow payoff is continuous and bounded on the compact state space $\mathcal{M}\times\mathcal{K}\times\mathcal{N}$, and the discount rate $\rho > 0$ is strictly positive. The Bellman operator is a contraction on the space of bounded value functions, and uniqueness of the fixed point follows from the Banach fixed-point theorem.

4.2. Model Performance

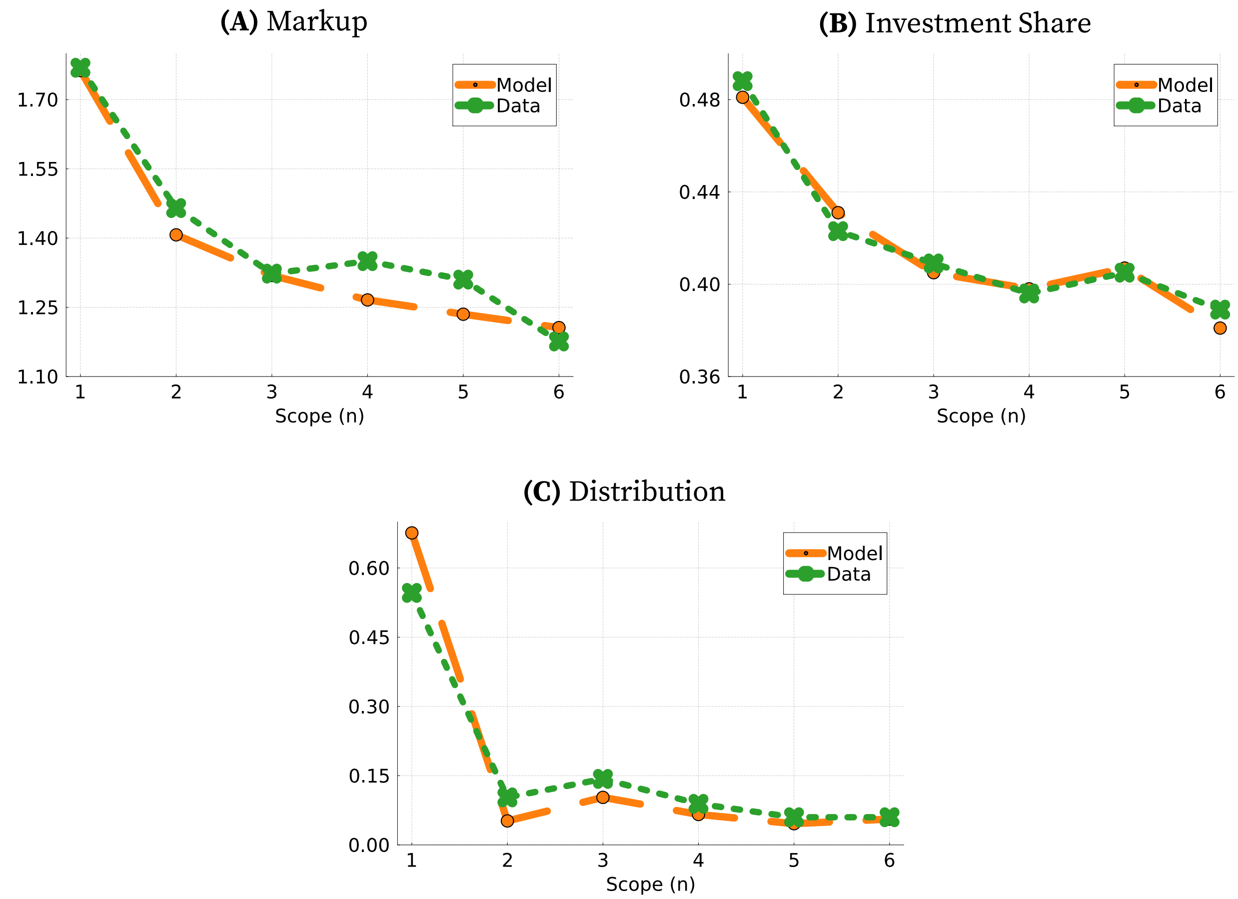

Figure 4. Targeted Moments: Model versus Data

Note: The green line represents the data and the orange line the model along the balanced growth path. The sample excludes firms in the utilities and finance sectors, as well as firms with missing or non-positive sale, employment, R&D and SG&A expenditures. Markup, the investment share, and the scope distribution are measured for the 2019 cross-section. The investment share is defined as the share of transferable investment in total investment, $I^T/(I^T + I^{\text{Emb}})$ (see Section 3.2), and is winsorized at the 95th percentile; markup is winsorized at the 90th percentile.

The model matches the targeted profiles closely along all three margins, reproducing both the levels and the gradients across scope. The markup schedule (Figure 4A) declines monotonically across scope levels, reflecting the span-of-control friction on managerial productivity, with only a slight underprediction at intermediate scopes where the data exhibit a small non-monotonic uptick that the smooth structural form cannot reproduce. A similar close fit holds for investment composition: the transferable investment share

(Figure 4B) tracks the data throughout, capturing both the sharp drop from $n = 1$ to $n = 2$, which reflects the substitution from transferable toward embedded intangibles as scope expands, and the subsequent flattening at higher scopes. The scope distribution (Figure 4C) replicates the pronounced concentration of mass at $n = 1$ and the rapid thinning of the distribution as scope expands.

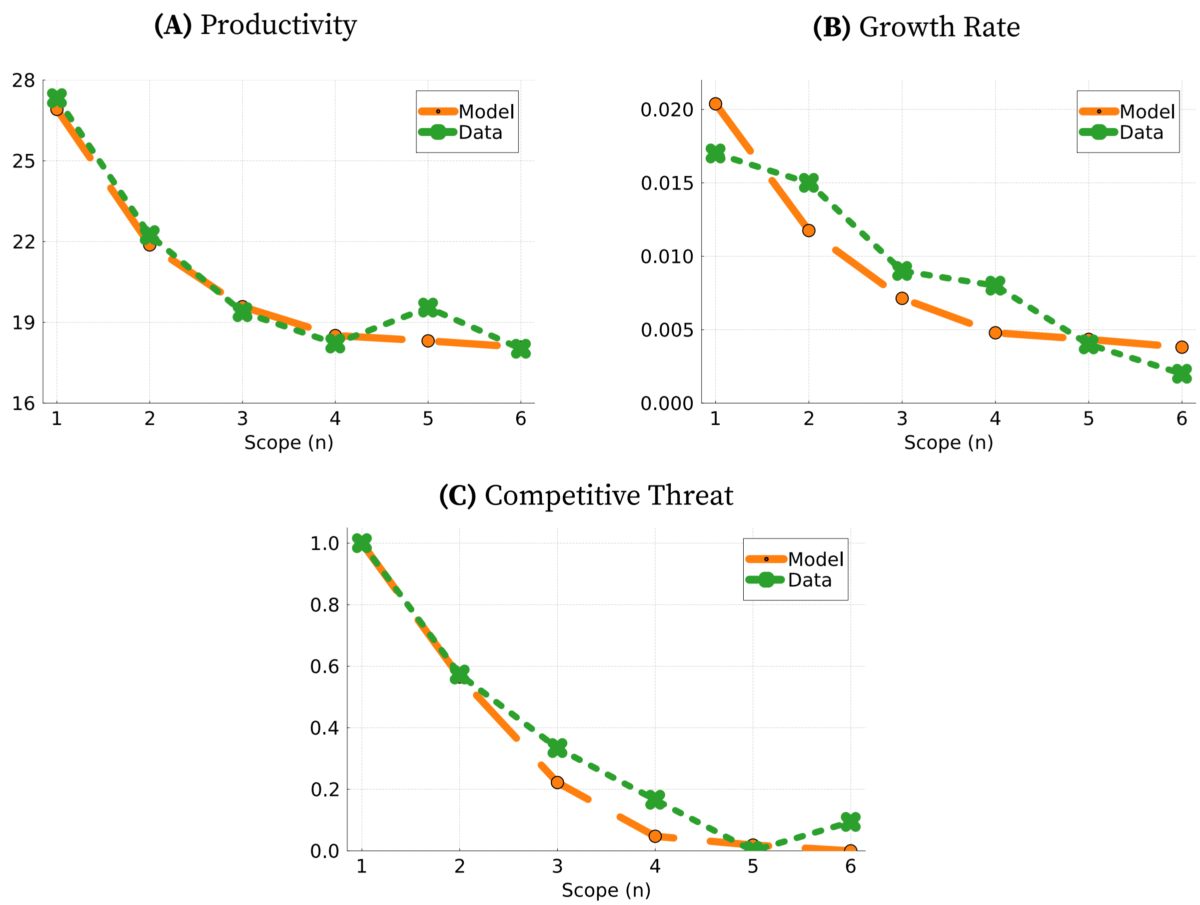

Figure 5. Untargeted Moments: Model versus Data

Note: The green line represents the data and the orange line the model along the balanced growth path. The sample excludes firms in the utilities and finance sectors, as well as firms with missing or non-positive sale, employment, R&D and SG&A expenditures. Productivity and competitive threat are measured for the 2019 cross-section and are described in Section 3.2. The growth rate is defined as the one-year log change in productivity, $\Delta \ln(\text{prod})_{i,t} = \ln(\text{prod}_{i,t}) - \ln(\text{prod}_{i,t-1})$, averaged over 2016–2019 and winsorized at the 0.05th and 95th percentiles and productivity is winsorized at the 95th percentile. Competitive threat, in both the model and data, is computed as the average at each scope level and rescaled via min–max normalization, $(\bar{x}(n) - \min \bar{x}(n))/(\max \bar{x}(n) - \min \bar{x}(n))$, which lies in $[0, 1]$, with higher values indicating greater competitive pressure.

Figure 5 evaluates the model along three untargeted dimensions: productivity, the productivity growth rate, and the competitive threat. Because these moments are not used in estimation, close correspondence between model and data provides direct evidence that the calibration recovers the structural mechanisms generating firm behavior rather than fitting statistical features specific to the targeted moments.

The productivity schedule (Figure 5A) is reproduced closely at every scope level, with the steep decline from $n = 1$ to $n = 3$ and the flattening at broader scopes both lying inside the data range. The only material divergence appears at $n = 5$, where the data exhibit a small uptick that the model's monotone schedule does not capture. The growth-rate profile (Figure 5B) inherits the same monotonic decline observed in the data, with the model producing a slightly smoother profile that nonetheless matches the empirical levels at the boundaries and remains close throughout the intermediate range. The competitive-threat profile (Figure 5C) delivers the model's most striking out-of-sample fit: the sharp decline from full incumbent exposure at $n = 1$ to near zero at broad scopes is reproduced almost exactly, supporting the channel through which embedded intangibles, and organizational capital in particular, generate scope-driven entry deterrence.

Together, the close fit along untargeted dimensions indicates that the calibrated parameters capture the underlying economic mechanisms rather than features specific to the targeted moments. The model is therefore a credible quantitative laboratory for the counterfactual analysis that follows.

4.3. Innovation Direction of Superstar Firms

To characterize how superstar firms allocate innovation effort between vertical and horizontal innovation, I construct a specialization index that summarizes the optimal direction at each (m, k) position for a given scope n:

$$S(m,k) \;=\; \frac{z^{\text{Ver}}(m,k) - \mathbb{P}_{s>s'}(m,k)\, z^{\text{Hor}}(m,k)}{z^{\text{Ver}}(m,k) + \mathbb{P}_{s>s'}(m,k)\, z^{\text{Hor}}(m,k)}. \tag{43}$$

The index lies in $[-1, 1]$, with $S = 1$ corresponding to pure vertical specialization, $S = -1$ to pure horizontal specialization, and $S = 0$ to indifference between the two directions. Because successful horizontal innovation requires the firm to defeat a randomly drawn incumbent in the target sector, the horizontal arrival rate z^{Hor} is weighted by the win probability $\mathbb{P}_{s>s'}$ to reflect the expected expansion rate.

Figure 6 plots S over the (m, k) grid at each of the six scope levels. Two patterns emerge. First, scope exerts a first-order effect: as firms expand from $n = 1$ to $n = 6$, the optimal direction shifts decisively toward vertical innovation across nearly the entire interior of the state space. Second, within each scope level, the direction varies with the firm's intangible composition, and the dependence is strongly nonlinear at narrow scope.

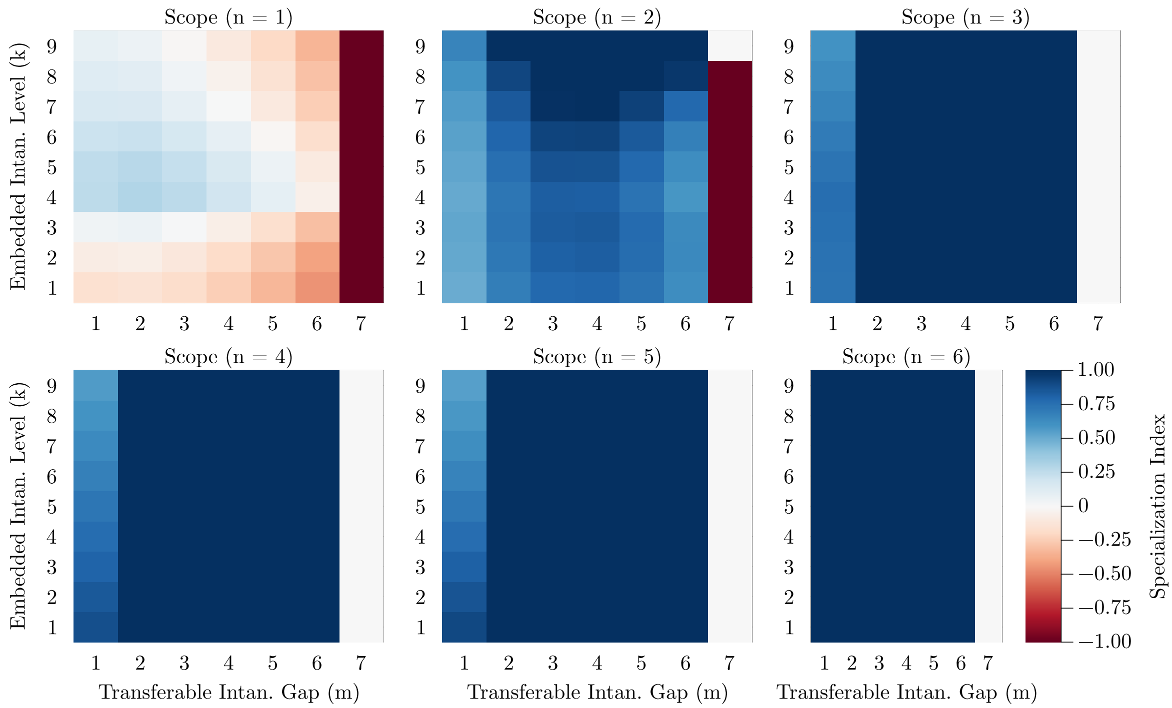

Figure 6. Specialization Index by Scope and Intangible Composition

Note: Each panel plots the specialization index $S(m, k)$ defined in equation (43) over the (m, k) grid for a given scope level n. Blue cells indicate specialization toward vertical innovation, red cells toward horizontal innovation, and white cells approximate indifference. Panels are arranged in order of increasing scope from $n = 1$ (top left) to $n = 6$ (bottom right).

At $n = 1$, the index displays substantial heterogeneity across the (m, k) grid. Three forces shape the pattern. The embedded stock k raises the marginal return to vertical innovation by increasing within-sector market share, so higher k tilts S upward. The transferable gap m reduces the marginal return to vertical improvement, since the firm is already closer to the within-sector frontier, so higher m tilts S downward. When both m and k are high, the free mobility of embedded intangibles across sectors reinforces the appeal of horizontal expansion, generating the deep red region in the upper-right portion of the panel. Vertical specialization is therefore strongest at high k and low m, while horizontal specialization dominates at high m regardless of k.

At $n = 2$, the same forces operate but the span-of-control friction begins to bind. Firms with high embedded stocks find that the market-share dilution from adding another sector outweighs the gains from horizontal expansion, shifting them firmly toward vertical innovation. The region of horizontal specialization shrinks substantially relative to $n = 1$, leaving most of the (m, k) interior with S close to $+1$.

For $n \geq 3$, the picture simplifies. The span-of-control friction is severe enough that horizontal expansion is unprofitable across nearly the entire interior of the (m, k) grid, and the index lies close to $+1$ at almost every state. Firms at higher scope therefore concentrate innovation effort on vertical quality improvement within their existing sectors, regardless of the precise composition of their intangible endowment.

A separate boundary effect appears in the rightmost column of every panel, where the transferable gap reaches its maximum $\bar{m}$. At this boundary, vertical innovation is infeasible because the firm cannot improve beyond the maximum quality lead. The implications for innovation direction depend on scope. At narrow scope ($n = 1, 2$), the only profitable alternative is horizontal expansion, and the boundary cells appear deep red. At broader scope ($n \geq 3$), the span-of-control friction also rules out horizontal expansion, leaving firms close to indifferent between the two directions and reducing total innovation effort. The boundary cells correspondingly transition from deep red to white as scope expands.

Together, Figure 6 establishes that innovation direction cannot be reduced to firm size alone. Scope governs the level of innovation incentives through market share and managerial costs, while the composition of the intangible endowment determines the direction of investment within each scope level. The interaction between these two margins generates the cross-scope heterogeneity in innovation strategies that the model is designed to explain.

5. Counterfactual Analysis and Misallocation

Two structural mechanisms shape misallocation in the calibrated model. The span-of-control friction reduces firm-level efficiency at every scope level, penalizing both horizontal expansion and the productivity of existing operations. Embedded intangibles operate in the opposite direction: by amplifying the incumbent's market share, they raise the marginal returns to vertical and horizontal innovation. Yet this firm-level efficiency gain comes with social cost. Embedded intangibles are firm-specific organizational knowledge that raises the incumbent's profitability without generating spillovers, and they erect entry barriers that shield superstars from creative destruction and redirect innovative resources toward barrier-building activity that is privately valuable but socially unproductive. The two mechanisms therefore stand in opposition at the aggregate level, motivating the counterfactual experiments below.

The first experiment shuts down each channel directly: the span-of-control technology by setting $n^{\alpha} = 1$ and the embedded-intangible accumulation by setting $k = 1$ for all firms. Removing the span-of-control friction shifts the firm distribution sharply toward two-sector and pushes innovation activity an order of magnitude above the baseline (Figure 7). Markups, however, track the baseline closely at narrow scope and diverge upward only at broad scope, where span-of-control friction was the binding constraint compressing pricing power. Eliminating embedded intangibles produces a nearly opposite pattern. The firm distribution remains close to the baseline, since span-of-control still penalizes broad scope, but markups fall across the entire scope distribution and competitive threat rises, reflecting the loss of organizational capital that had been deterring entry. Aggregate growth, however, stays close to the baseline. The first counterfactual experiment shows that span-of-control constraints discipline scope expansion, generate the declining markup gradient, and shape the growth profile across scope levels, while embedded intangibles govern competitive threats and markups. Neither channel alone is sufficient to account for the full set of empirical regularities.

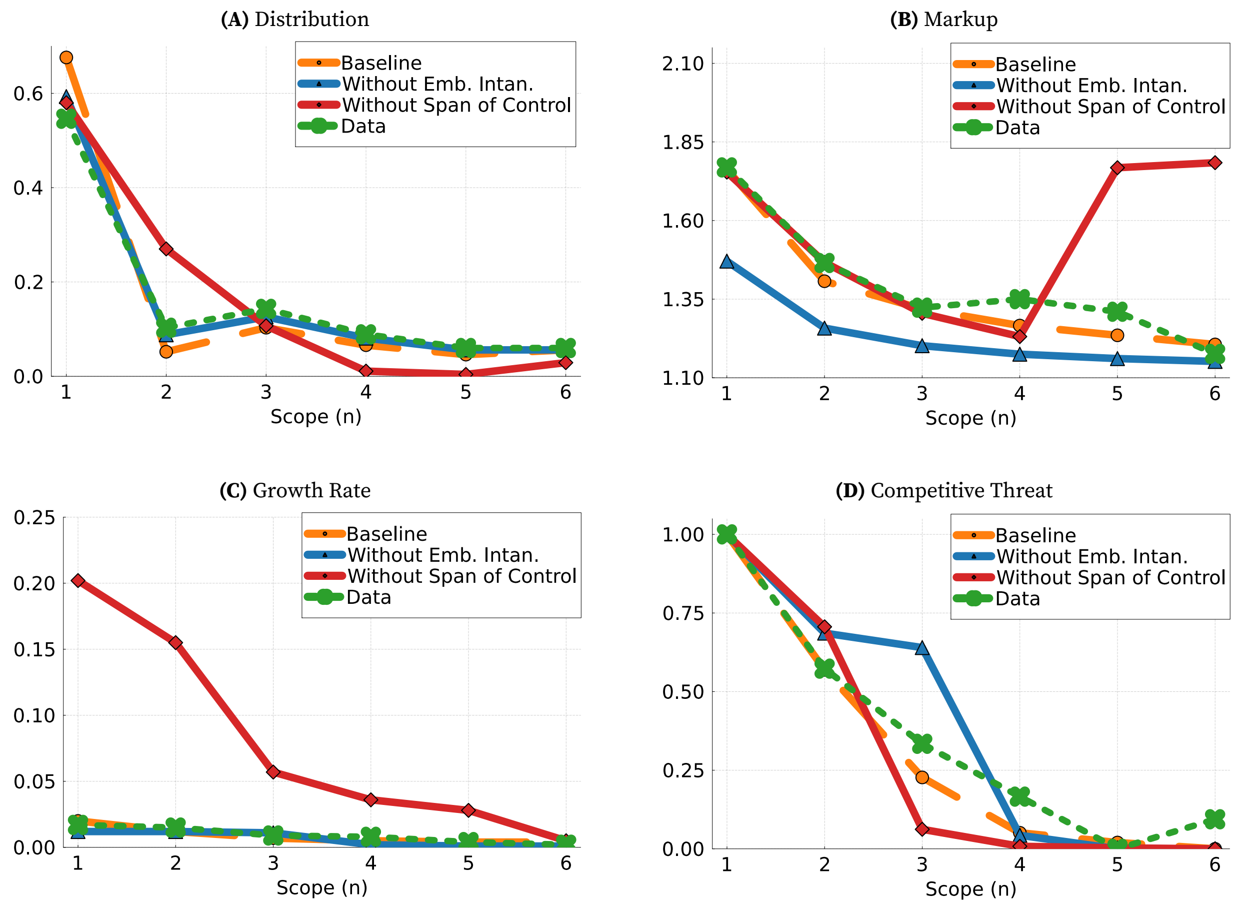

Figure 7. Counterfactual Analysis Under Different Cases

Note: The green line represents the data, whereas the orange line shows the model results along the balanced growth path. The red line corresponds to the case with the span-of-control channel shut down ($n^{\alpha} = 1$), while the blue line corresponds to the case with embedded intangibles shut down ($\bar{k} = 1$).The reported competitive threat measure is computed as the average at each scope level and then rescaled using a min–max normalization, $(x - \min x)/(\max x - \min x)$, to lie in $[0, 1]$, capturing relative variation across scope levels, with higher values indicating greater competitive pressure.

The second experiment isolates the contribution of entry barriers without shutting down the underlying technologies. The barriers operate through the displacement conditions that determine when a challenger or fringe entrant succeeds against an incumbent. I remove the scope and embedded-intangible barriers separately from these conditions while leaving the technologies themselves intact.[31] Figure 8 decomposes the aggregate growth rate into vertical innovation, horizontal innovation, and fringe entry across the

[31] *Counterfactual 1 (No n-barrier).* Setting $n_s = n_{s'} = 1$ in both displacement conditions reduces them to

$$\lambda(\theta^{k_s})^{\alpha} > (\theta^{k_{s'}})^{\alpha} \quad \text{and} \quad \lambda^{\bar{m}} > (\theta^{k_{s'}})^{\alpha}$$

.

three scenarios.

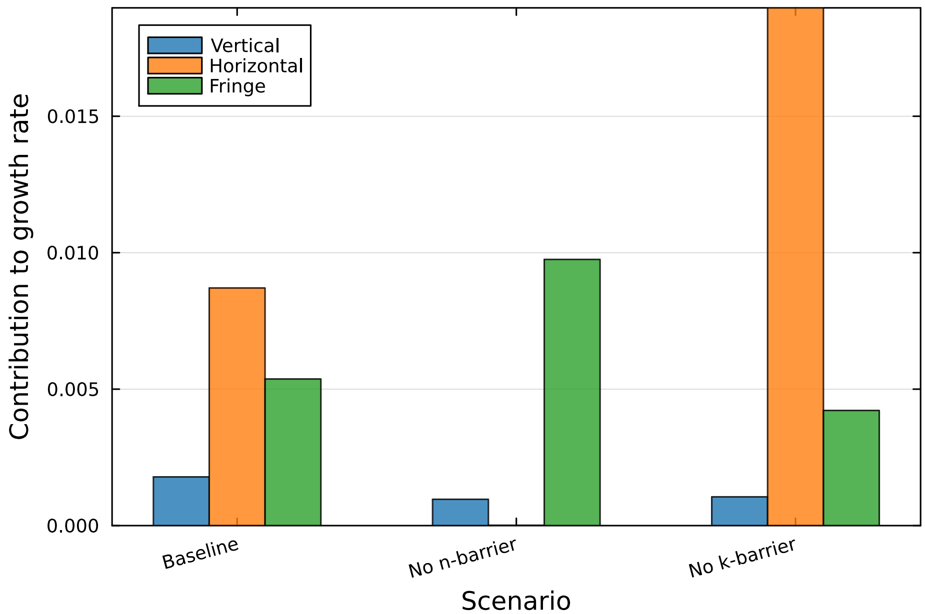

Figure 8. Growth Decomposition by Innovation Type and Entry Barrier

Note: The figure decomposes the aggregate growth rate into contributions from vertical innovation, horizontal innovation, and fringe entry across three scenarios: the baseline equilibrium, removal of the scope barrier (n-barrier), and removal of the embedded intangible barrier (k-barrier). Each bar reports the contribution of the respective innovation type to the balanced growth path rate g, as defined in equation (36).

The two barrier removals produce sharply asymmetric reallocations that trace directly to which competitor each barrier disciplines. Removing the n-barrier collapses incumbent horizontal innovation while fringe activity expands, leaving aggregate growth slightly below the baseline.[32] Scope is therefore the binding constraint protecting incumbents from fringe entry at broader scope. When the n-barrier is removed, every incumbent is treated as a single-sector firm in the displacement condition, fringe entry succeeds across the embedded distribution, and superstars optimally abandon horizontal innovation in favor of letting fringe activity drive growth. The decline in aggregate growth reflects the

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Counterfactual 2 (No k-barrier). Setting $k_s = k_{s'} = 1$ in both conditions reduces them to

$$\lambda(n_s)^{1-\alpha} > (n_{s'})^{1-\alpha} \quad \text{and} \quad \lambda^{\bar{m}} > (n_{s'})^{1-\alpha}.$$

Removing both barriers simultaneously yields results quantitatively similar to the n-barrier case and is omitted. Appendix Figure C2 reports both scenarios across scope.

[32]This is visible in Appendix Figure C1, where fringe entry mostly succeeds against narrow-scope incumbents at all embedded-intangible levels but is blocked once scope reaches $n > 3$.

appropriability mechanism of Aghion *et al.* (2005): when entry protection falls below a threshold, the private return to incumbent innovation collapses faster than entrant activity can compensate, placing the economy on the downward-sloping side of the inverted-U between competition and innovation.

Removing the k-barrier produces the opposite reallocation. Horizontal innovation surges well above its baseline level while fringe activity remains roughly unchanged, and aggregate growth rises substantially. In the baseline, low-k superstars are at a substantial disadvantage when attempting to expand into a sector held by a high-k incumbent, and they invest little in horizontal innovation. When the k-barrier is removed, the comparison no longer depends on embedded levels, previously disadvantaged superstars expand aggressively, and the resulting increase in incumbent scope offsets the loss of embedded protection that left fringe activity unchanged.

The two barriers therefore protect distinct innovation margins against different competitors. The n-barrier shields incumbents from fringe entry; removing it weakens incentives for horizontal expansion but stimulates innovation by fringe firms. The k-barrier, in contrast, shields incumbents from rival superstar challenges; this relaxation releases horizontal innovation among previously disadvantaged superstars but dampens fringe innovation.

6. Policy Implications

The counterfactual analysis in Section 5 establishes that directly removing scope and embedded-intangible barriers is counterproductive: barrier removal erodes the appropriability protection sustaining incumbent innovation, and the resulting contraction of vertical and horizontal investment more than offsets the expansion of fringe activity. Optimal policy must therefore work indirectly, discouraging barrier-building among broad-scope incumbents while redirecting investment toward the margins that raise aggregate productivity without generating firm-specific entry costs. The model's distinction between firm scope and firm size makes such a design feasible. A policy conditioned on scope rather than size targets exactly the firms whose horizontal expansion accumulates entry-deterring embedded intangibles, while leaving narrow-scope incumbents and fringe entrants free to innovate without distortion.[33]

I evaluate a family of progressive instruments parameterized by a threshold τ^*. Firms with scope above the threshold face a profit tax $\tau_{\text{scope}} = 0.10$ that compresses the value of incumbency and attenuates all three superstar innovation margins simultaneously, while firms at or below the threshold receive investment subsidies of magnitude 0.50 on one or combination of $\{\tau_{\text{ver}}, \tau_{\text{hor}}, \tau_{\text{emb}}, \tau_f\}$,[34] each acting as a multiplicative cost reduction on the targeted margin alone.[35] A profit tax operates directly on the appropriated rent flow and so produces a behavioral response of magnitude τ_{scope} at the value-function level; an investment subsidy, by contrast, lowers the marginal cost of z but the firm re-optimizes along the convex investment schedule, so a substantial fraction of the rate reduction is absorbed by higher chosen z rather than translating into an equally large change in value.[36]

The threshold τ^* governs the progressivity of the design, following Berlingieri *et al.* (2025). At one extreme ($\tau^* = 0$), the scope tax applies to all incumbent firms with no offsetting subsidy: a *flat tax*. At the other extreme ($\tau^* = \bar{n}$), every firm receives a subsidy with no taxation: a *flat subsidy*. Between the endpoints, the tax base narrows progressively toward broader-scope firms while subsidy eligibility widens toward narrower-scope firms and fringe entrants. The government budget is balanced through lump-sum transfers to

[33] Appendix B.7 derives the social planner's problem, and Appendix B.8 presents the Pigouvian tax derivation.

[34] The rate asymmetry reflects the disparity between the two fiscal bases. Equilibrium incumbent profit flows in the calibrated stationary distribution exceed equilibrium investment expenditures by roughly an order of magnitude, so identical statutory rates would mechanically generate a much larger tax extraction than subsidy injection.

[35] The scope tax modifies the profit flow in the superstar value function (26) to $(1 - \tau_{\text{scope}})\,\pi(m, k, n)$ for firms with $\tau(n) > \tau^*$. The investment subsidies enter as multiplicative cost shifters in equation (8):

$$(1 - \tau_{\text{ver}})\,\gamma^{\text{Ver}}(z^{\text{Ver}})^{\vartheta^{\text{Ver}}}, \quad (1 - \tau_{\text{hor}})\,\gamma^{\text{Hor}}(z^{\text{Hor}})^{\vartheta^{\text{Hor}}}, \quad (1 - \tau_{\text{emb}})\,\gamma^{\text{Emb}}(z^{\text{Emb}})^{\vartheta^{\text{Emb}}},$$

each acting on the targeted margin alone. The fringe subsidy reduces fringe innovation costs in (27) to $(1 - \tau_f)\,\gamma^f(z_f)^{\vartheta^f}$.

[36] The full grid of fifteen non-empty instrument combinations is reported in Appendix Figures C3 and C4; the main text focuses on the three designs most relevant to correcting the identified misallocation.

the representative household.

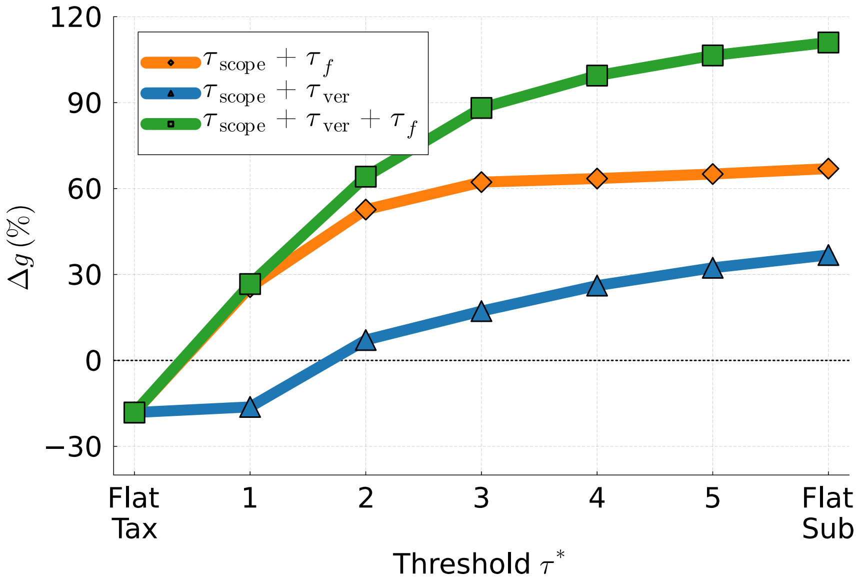

Figure 9. Flat and Progressive Policies: Effect on the Growth Rate

Note: Each line reports the percentage change in the aggregate balanced-growth-path rate Δg (%) relative to the baseline for a given subsidy combination paired with the scope profit tax. For a given threshold τ^*, superstar firms with $\tau(n) > \tau^*$ face the scope profit tax $\tau_{\text{scope}} = 0.10$, while firms with $\tau(n) \leq \tau^*$ receive cost reductions of magnitude 0.50 on the indicated investment margins, each entering as $(1 - \tau)$ on the respective cost; τ_f applies to fringe entrants at all scope levels. The left endpoint ($\tau^* = 0$) corresponds to a flat tax on all firms with no subsidy; the right endpoint ($\tau^* = \bar{n}$) corresponds to a flat subsidy with no tax.

Figure 9 reports the percentage change in the aggregate balanced-growth-path rate Δg across the threshold spectrum for three representative designs: the joint scheme $\tau_{\text{scope}} + \tau_{\text{ver}} + \tau_f$ and its two restrictions $\tau_{\text{scope}} + \tau_f$ and $\tau_{\text{scope}} + \tau_{\text{ver}}$. All three deliver an identical -18% growth loss at the flat-tax extreme ($\tau^* = 0$), where subsidy eligibility is empty and every configuration reduces to broad-based scope taxation. As the threshold widens, growth rises monotonically for each design, but the joint scheme dominates at every interior threshold, reaching $\Delta g \approx +111\%$ at the flat-subsidy limit. The fringe subsidy is the single most important contributor: $\tau_{\text{scope}} + \tau_f$ delivers more than twice the growth of $\tau_{\text{scope}} + \tau_{\text{ver}}$ across most thresholds.

The consumption-equivalent welfare ranking, reported in Figure 10, reverses this conclusion sharply. The joint design $\tau_{\text{scope}}(n > 1) + \tau_{\text{ver}}(n \leq 1) + \tau_f$ at $\tau^* = 1$ delivers the unique welfare optimum, $\Delta\Omega \approx +14\%$ with associated growth $\Delta g \approx +27\%$; the restriction $\tau_{\text{scope}} + \tau_f$ at the same threshold achieves an essentially indistinguishable $\Delta\Omega \approx +13\%$. At the flat-subsidy limit, by contrast, the joint design's growth gain of $+111\%$ delivers only $\Delta\Omega \approx +5\%$ — less than half the welfare attained at $\tau^* = 1$. The integrated consumption-equivalent metric makes visible what the asymptotic growth rate conceals: additional investment cost incurred at broader subsidy thresholds crowds out current consumption faster than it raises long-run output.[37]

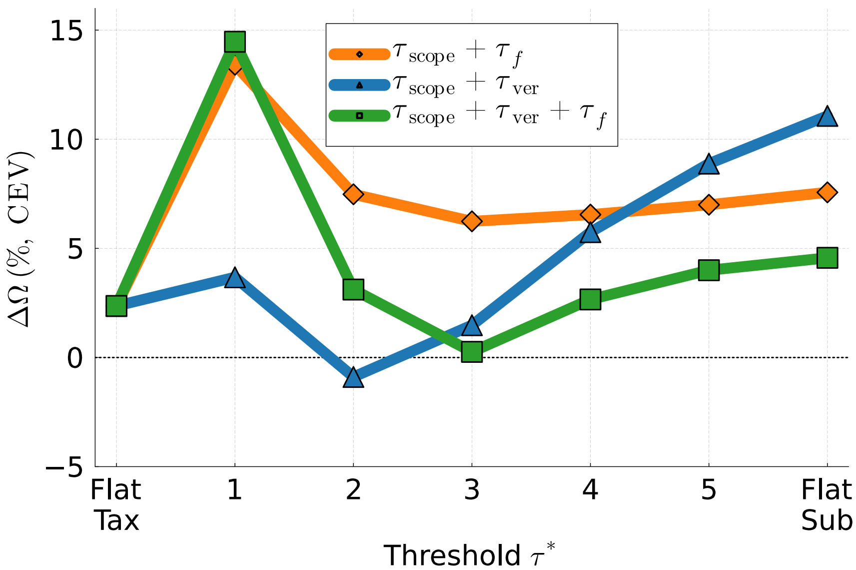

Figure 10. Flat and Progressive Policies: Effect on Welfare

Note: Each line reports the consumption-equivalent welfare gain $\Delta\Omega$ (%, CEV) relative to the baseline for a given subsidy combination paired with the scope profit tax. For a given threshold τ^*, superstar firms with $\tau(n) > \tau^*$ face the scope profit tax $\tau_{\text{scope}} = 0.10$, while firms with $\tau(n) \leq \tau^*$ receive cost reductions of magnitude 0.50 on the indicated investment margins, each entering as $(1-\tau)$ on the respective cost; τ_f applies to fringe entrants at all scope levels. The left endpoint ($\tau^* = 0$) corresponds to a flat tax on all firms with no subsidy; the right endpoint ($\tau^* = \bar{n}$) corresponds to a flat subsidy with no tax. See Appendix B.6 for the welfare accounting procedure.

The welfare-maximizing design requires three structural features.[38] The scope tax must apply to firms with $n > 1$, internalizing the entry-barrier externality at its source. The only incumbent subsidy must be on vertical innovation, restricted to the narrowest scope tier — firms whose innovation does not accumulate entry-deterring embedded capital. And the fringe subsidy must operate at all scope levels, restoring the creative destruction channel that aggregate productivity depends on. The third line-plot design,

[37]The disconnect is far more extreme on the full instrument grid (Appendix Figure C4). The maximally permissive all-instrument flat subsidy that maximizes growth simultaneously produces the worst welfare outcome on the grid.

[38]See Appendix B.6 for the welfare accounting procedure.

$\tau_{\text{scope}} + \tau_{\text{ver}}$ without τ_f, is welfare-monotone in the threshold but reaches only $\Delta\Omega \approx +11\%$ at the flat-subsidy limit and produces the lowest growth of the three ($\Delta g \approx +37\%$) — confirming that a coherent welfare-improving design requires the fringe subsidy paired with the scope tax.

7. Conclusion

This paper develops a unified endogenous growth model in which the direction of innovation emerges from firms' expansion decisions. Scope expansion introduces span-of-control frictions that compress managerial productivity, leading multi-sector firms to reallocate investment from transferable intangibles, which generate knowledge spillovers, toward embedded intangibles, which provide firm-specific competitive advantages. Joint accumulation of scope and embedded intangibles raises entry barriers and suppresses creative destruction, generating a wedge between private and social returns to innovation that instruments conditioned on firm size cannot address.

Calibrating the model to firm-level evidence on intangible composition, markups, productivity, and scope, I show that scope-conditional progressive policies improve both aggregate growth and welfare, whereas flat designs improve neither. The welfare-maximizing instrument taxes incumbents operating in more than one sector and pairs this levy with a vertical-innovation subsidy restricted to single-sector incumbents and a fringe-entry subsidy available at all scope tiers, delivering a 27 percent increase in the balanced-growth rate and a 14 percent consumption-equivalent welfare gain. Aligning innovation incentives with social returns therefore requires conditioning policy on scope and intangible composition rather than on firm size, which aggregates over precisely the margins that determine whether innovation is socially productive or barrier-reinforcing.or barrier-reinforcing.

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Appendices

A. Empirical Appendix

A.1. Dataset

Table A1. Example Firms Segment in Compustat Segment Dataset

Company	**Segments**
TOYOTA MOTOR CORP	Financial Services Automotive All Other
PROCTER & GAMBLE CO	Health Care Grooming Corporate Beauty Baby, Feminine & Family Care Fabric & Home Care
TESLA INC	Energy Generation & Storage Automotive

A.2. Measurement Details

A.2.1. Production Function Estimation

Sample and specification. The estimation sample covers 1990–2019 and excludes the finance and utilities sectors. All estimations are conducted separately for each two-digit NAICS industry to allow technology to vary across industries. The gross-output Cobb-Douglas production function is

$$y_{it} = \beta_l l_{it} + \beta_k k_{it} + \beta_m m_{it} + \omega_{it} + \epsilon_{it}, \tag{44}$$

where y_{it} is log gross output (`sale`), l_{it} is log labor (`emp`), k_{it} is log capital (`ppent`), and m_{it} is log intermediate inputs (`cogs`). The term ω_{it} is the unobserved firm-level productivity shock observed by the firm before making input decisions, and ϵ_{it} is an i.i.d. measurement error orthogonal to inputs.

The sample is restricted to observations satisfying `sale` > 0, `ppent` > 0, `cogs` > 0, `emp` > 0, `capx` > 0, and `cogs` $<$ `sale`, where the final restriction ensures positive gross profits. All nominal variables are deflated to real terms prior to estimation: output by the GDP deflator, intermediate inputs by the Producer Price Index, and capital stock and capital expenditure by the Gross Private Domestic Investment deflator.[39]

Baseline: Olley and Pakes (1996). The baseline estimator follows the two-stage proxy-variable procedure of Olley and Pakes (1996). Under the assumption that capital investment is strictly increasing in productivity conditional on capital, the firm's investment policy function $i_{it} = f_t(\omega_{it}, k_{it})$ is invertible in ω_{it}, yielding $\omega_{it} = h_t(k_{it}, i_{it})$, where i_{it} is log capital expenditure (`capx`).

First stage. Substituting the inversion into the production function gives the first-stage estimating equation

$$y_{it} = \beta_l l_{it} + \beta_m m_{it} + \phi_t(k_{it}, i_{it}) + \epsilon_{it}, \tag{45}$$

where $\phi_t(k_{it}, i_{it}) \equiv \beta_k k_{it} + h_t(k_{it}, i_{it})$ is approximated by a third-order polynomial in (k_{it}, i_{it}). The first stage identifies β_l, β_m, and the composite function ϕ_t.

Second stage. The coefficient β_k is identified by exploiting the Markov property of productivity. Under $\omega_{it} = g(\omega_{i,t-1}) + \xi_{it}$, the productivity innovation ξ_{it} is orthogonal to capital chosen at $t-1$, yielding the moment condition

$$\mathbb{E}[\xi_{it}(\beta_k) \cdot k_{it}] = 0, \tag{46}$$

where $\xi_{it}(\beta_k)$ is constructed from the first-stage residual of the productivity process. Selection bias from firm exit is corrected by including the estimated survival probability in

[39] GDP deflator: `https://fred.stlouisfed.org/series/A191RD3A086NBEA`. PPI: `https://fred.stlouisfed.org/series/WPUID61`. Investment deflator: `https://fred.stlouisfed.org/series/A006RD3A086NBEA`.

the second-stage moment, as in Olley and Pakes (1996).

Robustness: Ackerberg *et al.* (2015). Ackerberg *et al.* (2015) identify the first stage estimates only the composite function $\phi_t(l_{it}, k_{it}, m_{it}, i_{it})$ nonparametrically, and β_l, β_k, and β_m are identified jointly in the second stage via moment conditions on the productivity innovation, using lagged values of variable inputs as instruments to exploit the timing assumption that these inputs are determined before ω_{it} realizes.

Robustness: Gandhi *et al.* (2020). Gandhi *et al.* (2020) identify the output elasticity of the flexible input from the firm's static cost-minimization condition, without imposing functional form restrictions on the production function. Under perfect competition in intermediate input markets, cost minimization implies that the output elasticity of intermediates equals the revenue share of intermediate input expenditure, after correcting for measurement error in observed output:

$$\frac{\partial \ln Q_{it}}{\partial \ln M_{it}} = \frac{P_{it}^M M_{it}}{P_{it} Q_{it}} \cdot \mathcal{E}(\epsilon_{it}), \tag{47}$$

where $\mathcal{E}(\epsilon_{it}) \equiv \mathbb{E}[e^{\epsilon_{it}}]$ is the measurement-error correction. This share equation identifies the flexible-input elasticity in the first stage without functional-form restrictions on the production function. The remaining input coefficients are recovered in the second stage using moment conditions on the productivity innovation, analogous to the OP and ACF second stages.

A.3. Additional Figures and Empirical Results

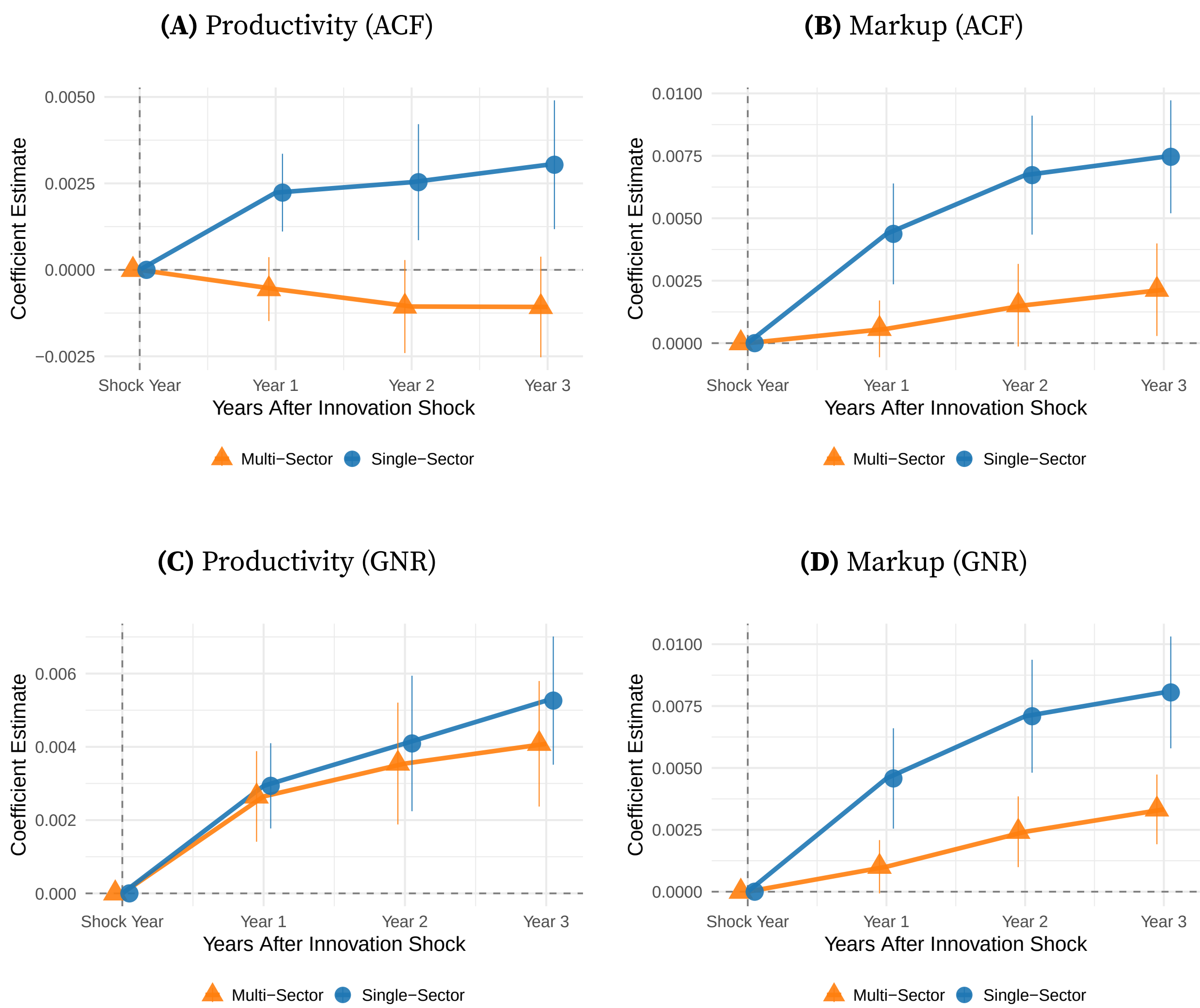


Figure A1. Productivity and Markup Responses to Innovation Shocks: ACF and GNR

Note: Panels (A) and (B) report estimates based on Ackerberg *et al.* (2015); Panels (C) and (D) report estimates based on Gandhi *et al.* (2020). The innovation shock is measured as $\log(1+\text{PatentValue}_{it})$, preserving firms with zero patent grants in a given year. Outcome variables and controls are measured in $\log(\cdot)$. The sample excludes utilities and finance sectors, as well as firms with missing or non-positive R&D, SG&A, employment, and sales. All variables cover the period 1990–2019. Vertical bars denote 95% confidence intervals based on standard errors clustered at the industry-year level.

Table A2. Regression Results

Panel A: *Markup* and *Productivity*

	Pooled OLS		Two-way FE	
	(2) *Markup*	(3) *Productivity*	(6) *Markup*	(7) *Productivity*
Production Lines	-0.038^{***}	-0.056^{***}	-0.041^{***}	-0.044^{***}
	(0.004)	(0.008)	(0.004)	(0.003)
Num. Obs.	40,008	40,008	40,008	40,008
Adj. R^2	0.268	0.197	0.279	0.833
Covariates	Yes	Yes	Yes	Yes
FE: *Year*	No	No	Yes	Yes
FE: *Industry*	No	No	Yes	Yes

Panel B: *Investment Ratio* and *Fluidity*

	Pooled OLS		Two-way FE	
	(1) *Transferable/Embedded*	(4) *Comp. Threat*	(5) *Transferable/Embedded*	(8) *Comp. Threat*
Production Lines	-0.011^{***}	-0.037^{***}	-0.014^{***}	-0.017^{**}
	(0.002)	(0.006)	(0.002)	(0.006)
Num. Obs.	42,831	31,318	42,831	31,318
Adj. R^2	0.977	0.208	0.978	0.305
Covariates	Yes	Yes	Yes	Yes
FE: *Year*	No	No	Yes	Yes
FE: *Industry*	No	No	Yes	Yes

Notes: Each column reports coefficients from a separate regression. Standard errors are clustered by *firm id* (gvkey) in parentheses. Significance levels: $^{*}p < 0.05$, $^{**}p < 0.01$, $^{***}p < 0.001$. Columns 1–4: Pooled OLS specifications; Columns 5–8: Two-way fixed effects (year and industry). Covariates include sale, xrd, and xsga.

Table A3. Summary Statistics

Panel A: Compustat Fundamentals

Variable	N	Mean	SD	P25	Median	P75
Sale	71,139	3,398,985	17,129,640	22,203	121,055	846,755
Emp	71,139	9,814	37,290	135	597	3,702
AT	71,136	4,502,256	23,855,832	26,457	142,066	981,467
XRD	71,139	147,233	746,922	1,589	7,513	35,977
XSGA	71,139	641,985	2,651,924	11,640	44,070	208,185
Productivity (OLS)	65,211	20.416	23.416	7.861	11.106	24.321
Markup (OLS)	64,591	1.543	1.042	0.988	1.210	1.665
Productivity (ACF)	51,847	30.191	49.551	6.410	8.391	36.216
Markup (ACF)	45,510	1.275	0.768	0.885	1.080	1.417
Productivity (GNR)	65,843	11.888	22.762	0.880	1.562	7.911
Markup (GNR)	64,647	1.291	0.914	0.826	1.014	1.382
Competitive Threat	51,975	6.781	3.211	4.442	6.370	8.619

Panel B: Compustat–Segment Merged (including ADR)

Variable	N	Mean	SD	P25	Median	P75
Sale	43,007	3,597,348	16,753,747	29,041	166,258	1,120,375
Emp	43,007	10,466	37,694	154	708	4,482
AT	43,007	5,101,380	26,098,750	37,743	211,989	1,398,987
XRD	43,007	180,872	867,769	2,154	10,697	49,903
XSGA	43,007	721,384	2,891,804	14,998	59,082	266,119
Productivity (OLS)	40,008	21.055	24.665	8.054	11.264	25.035
Markup (OLS)	39,603	1.549	1.056	0.986	1.205	1.674
Productivity (ACF)	31,937	31.702	53.825	6.469	8.471	36.590
Markup (ACF)	28,218	1.287	0.785	0.889	1.082	1.423
Productivity (GNR)	40,404	11.991	23.298	0.891	1.549	7.952
Markup (GNR)	39,680	1.300	0.935	0.825	1.011	1.388
Competitive Threat	31,318	6.476	3.120	4.203	6.020	8.183

Panel C: Compustat–Segment Merged (excluding ADR)

Variable	N	Mean	SD	P25	Median	P75
Sale	35,964	2,171,465	10,468,548	25,104	135,367	826,899
Emp	35,964	6,440	23,946	133	571	3,212
AT	35,964	3,083,646	19,749,630	30,984	167,030	977,718
XRD	35,964	127,011	736,443	1,813	8,868	39,625
XSGA	35,964	476,449	2,256,258	13,128	50,168	208,927
Productivity (OLS)	33,376	20.556	24.630	7.890	11.082	23.682
Markup (OLS)	33,035	1.545	1.048	0.986	1.207	1.673
Productivity (ACF)	26,866	31.199	53.967	6.398	8.439	34.253
Markup (ACF)	23,712	1.295	0.795	0.895	1.089	1.434
Productivity (GNR)	33,706	11.672	23.026	0.883	1.473	7.212
Markup (GNR)	33,128	1.298	0.928	0.826	1.014	1.391
Competitive Threat	30,097	6.451	3.104	4.189	6.005	8.156

Notes: This table reports summary statistics for the three estimation samples. Panel A covers Compustat Fundamentals Annual restricted to firm-years with positive sales, employment, R&D (XRD), and SG&A (XSGA). Panel B merges Panel A with the Compustat Segment database, retaining all countries. Panel C restricts Panel B to U.S.-incorporated firms (*fic* = USA).

Table A4. Sector-Level Elasticities: OP, ACF, and GNR

		OP				*ACF*				*GNR*			
NAICS	Sector	β_M	β_K	β_L	RTS	β_M	β_K	β_L	RTS	β_M	β_K	β_L	RTS
11	Agriculture, Forestry & Fishing	0.834	0.032	0.100	0.966	0.612	0.146	0.137	0.894	0.676	0.209	0.148	1.030
21	Mining & Oil and Gas Extraction	0.719	0.118	0.050	0.887	0.692	0.280	0.001	0.972	0.500	0.386	0.111	0.997
23	Construction	0.931	0.023	0.012	0.966	—	—	—	—	0.826	0.018	0.131	0.975
31	Food & Beverage Manufacturing	0.853	0.029	0.066	0.947	—	—	—	—	0.659	0.148	0.189	0.996
32	Chemical & Paper Manufacturing	0.612	0.019	0.275	0.907	0.351	0.094	0.501	0.946	0.489	0.106	0.399	0.994
33	Machinery & Computer Manufacturing	0.724	0.027	0.184	0.935	0.692	0.076	0.199	0.967	0.617	0.091	0.315	1.020
42	Wholesale Trade	0.832	0.046	0.086	0.963	0.504	0.065	0.330	0.900	0.790	0.054	0.160	1.000
44	Retail Trade I	0.782	0.043	0.106	0.931	—	—	—	—	0.728	0.120	0.203	1.050
45	Retail Trade II	0.780	0.046	0.064	0.889	0.466	0.094	0.290	0.850	0.681	0.154	0.207	1.040
48	Transportation & Warehousing	—	—	—	—	0.628	0.120	0.113	0.860	0.755	0.196	0.065	1.020
51	Information	0.490	0.059	0.336	0.884	—	—	—	—	0.354	0.128	0.468	0.951
53	Real Estate & Rental	—	—	—	—	—	—	—	—	0.363	0.188	0.306	0.857
54	Professional & Technical Services	0.634	0.054	0.183	0.871	0.119	0.175	0.583	0.877	0.580	0.132	0.310	1.020
56	Administrative & Support Services	0.766	0.053	0.078	0.897	—	—	—	—	0.655	0.170	0.185	1.010
61	Educational Services	0.708	0.109	0.043	0.860	0.446	0.137	0.281	0.864	0.486	0.211	0.287	0.984
62	Health Care & Social Assistance	0.819	0.052	0.011	0.882	0.396	0.122	0.317	0.834	0.708	0.110	0.117	0.935
71	Arts, Entertainment & Recreation	0.871	0.054	0.023	0.948	0.424	0.246	0.177	0.847	0.649	0.179	0.135	0.963
72	Accommodation & Food Services	0.819	0.095	0.021	0.935	0.548	0.201	0.185	0.934	0.739	0.197	0.110	1.040
81	Other Services	—	—	—	—	0.796	0.053	0.105	0.954	0.665	0.102	0.139	0.905
99	Nonclassifiable Establishments	0.748	0.007	0.159	0.915	0.418	0.084	0.393	0.894	0.617	0.074	0.251	0.942

Notes: β_M, β_K, and β_L denote elasticities with respect to materials (flexible input), capital, and labor (both dynamic inputs), respectively. RTS $\equiv \beta_M + \beta_K + \beta_L$ is the implied returns to scale. OP refers to olleypakes1996; ACF refers to ackerberg2015identification; GNR refers to gandhi2020identification. Missing values (—) indicate that the estimation did not produce valid positive elasticities or RTS $\notin [0.8, 1.2]$.

B. Model Appendix

B.1. Intermediate Good Sector Demand

The cost minimization problem for the intermediate sector j is

$$\min_{y_{sjt}, y_{fjt}} \quad p_{sjt} y_{sjt} + p_{fjt} y_{fjt} \quad \text{s.t.} \quad y_{jt} = \left(\Xi(e_{st}) y_{sjt}^{\varepsilon} + (1 - \Xi(e_{st})) y_{fjt}^{\varepsilon} \right)^{1/\varepsilon}. \tag{48}$$

The first-order condition for y_{sjt} is

$$p_{sjt} = \lambda \, \Xi(e_{st}) \, y_{jt}^{1-\varepsilon} \, y_{sjt}^{\varepsilon-1}, \tag{49}$$

where λ is the Lagrange multiplier. Solving the first-order conditions for y_{sjt} and y_{fjt} and substituting into the production constraint identifies λ as the dual CES price index:

$$p_j \;\equiv\; \lambda \;=\; \left(\Xi(e_{st})^{-1/(\varepsilon-1)} p_{sjt}^{\varepsilon/(\varepsilon-1)} + (1 - \Xi(e_{st}))^{-1/(\varepsilon-1)} p_{fjt}^{\varepsilon/(\varepsilon-1)} \right)^{(\varepsilon-1)/\varepsilon}. \tag{50}$$

Substituting $\lambda = p_j$ back into the first-order condition and using $p_j y_{jt} = Y_t$ (which follows from the final-goods producer's optimization) yields the demand function

$$y_{sjt} = p_j^{\varepsilon/(1-\varepsilon)} \, \Xi(e_{st})^{1/(1-\varepsilon)} \, p_{sjt}^{1/(\varepsilon-1)} \, Y_t. \tag{51}$$

B.2. Superstar Firm Maximization Problem: Bertrand Competition

The superstar firm competes à la Bertrand with a continuum of fringe firms. Its profit maximization problem in industry j is:

$$\max_{p_{sjt}} \, (p_{sjt} - MC_{sjt}) \, y_{sjt} \quad s.t. \quad y_{sjt} = p_j^{\frac{\varepsilon}{1-\varepsilon}} \, \Xi(e_s)^{\frac{1}{1-\varepsilon}} p_{sjt}^{\frac{1}{\varepsilon-1}} Y_t. \tag{52}$$

Substituting the expression for p_j into the demand function, the objective expands to:

$$\max_{p_{sjt}} \left[p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} \, \Xi(e_s)^{\frac{1}{1-\varepsilon}} Y_t \left(\Xi(e_s)^{\frac{-1}{\varepsilon-1}} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} + (1-\Xi(e_s)) p_{fjt}^{\frac{\varepsilon}{\varepsilon-1}} \right)^{-1} \right. \tag{53}$$

$$\left. - MC_{sjt} \left(\Xi(e_s)^{\frac{-1}{\varepsilon-1}} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} + (1-\Xi(e_s)) p_{fjt}^{\frac{\varepsilon}{\varepsilon-1}} \right)^{-1} p_{sjt}^{\frac{1}{\varepsilon-1}} \, \Xi(e_s)^{\frac{1}{1-\varepsilon}} Y_t \right]. \tag{54}$$

Computing the derivative $\frac{\partial \pi}{\partial p_{sjt}}$ and factoring common terms gives:

$$= Y_t \, \Xi(e_s)^{\frac{1}{1-\varepsilon}} \left(\Xi(e_s)^{\frac{-1}{\varepsilon-1}} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} + (1-\Xi(e_s)) \, p_{fjt}^{\frac{\varepsilon}{\varepsilon-1}} \right)^{-1} \times \left\{ \left[\frac{\varepsilon}{\varepsilon-1} p_{sjt}^{\frac{1}{\varepsilon-1}} - MC_{sjt} \frac{1}{\varepsilon-1} p_{sjt}^{\frac{2-\varepsilon}{\varepsilon-1}} \right] \right.$$
$$\left. - \left(p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} - MC_{sjt} \, p_{sjt}^{\frac{1}{\varepsilon-1}} \right) \left(\Xi(e_s)^{\frac{-1}{\varepsilon-1}} \frac{\varepsilon}{\varepsilon-1} p_{sjt}^{\frac{1}{\varepsilon-1}} \left(\Xi(e_s)^{\frac{-1}{\varepsilon-1}} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} + (1-\Xi(e_s)) \, p_{fjt}^{\frac{\varepsilon}{\varepsilon-1}} \right)^{-1} \right) \right\}. \tag{55}$$

Multiplying through by p_{sjt} and substituting the market share ϕ_{sjt} from (19) yields:

$$\left(p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} - MC_{sjt} p_{sjt}^{\frac{1}{\varepsilon-1}} \right) \left(\phi_{sjt} \frac{\varepsilon}{\varepsilon-1} \right) = \left[\frac{\varepsilon}{\varepsilon-1} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} - MC_{sjt} \frac{1}{\varepsilon-1} p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}} \right]. \tag{56}$$

Dividing both sides by $p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}}$ and then by $\frac{\varepsilon}{\varepsilon-1}$, and rearranging:

$$\varepsilon(1-\phi_{sjt}) = \frac{MC_{sjt}}{p_{sjt}} (1-\varepsilon\phi_{sjt}). \tag{57}$$

Solving for the optimal price gives:

$$p_{sjt} = \frac{1-\varepsilon\phi_{sjt}}{\varepsilon(1-\phi_{sjt})} \, MC_{sjt}. \tag{58}$$

To determine optimal labor demand, equating output (6) with demand (51) yields

$$q_{sjt} \, \psi(e_s, n_s) \, l_{sjt} = p_j^{\frac{\varepsilon}{1-\varepsilon}} \Xi(e_s)^{\frac{1}{1-\varepsilon}} p_{sjt}^{\frac{1}{\varepsilon-1}} Y_t. \tag{59}$$

Multiplying both sides by p_{sjt} and dividing by w_t gives

$$p_{sjt}\,\underbrace{\frac{q_{sjt}\,\psi(e_s,n_s)}{w_t}}_{\text{inverse } MC_{sjt}}\,l_{sjt} = \underbrace{p_j^{\frac{\varepsilon}{1-\varepsilon}}\,\Xi(e_s)^{\frac{1}{1-\varepsilon}}\,p_{sjt}^{\frac{\varepsilon}{\varepsilon-1}}}_{\phi_{sjt}}\,\underbrace{\frac{Y_t}{w_t}}_{\omega_t^{-1}}, \tag{60}$$

which leads directly to equation (24).

B.3. Superstar Firm Maximization Problem: Cournot Competition

Under Cournot competition, the superstar firm and the aggregate fringe compete in quantities. The superstar solves

$$\max_{y_{sjt}}\;(p_{sjt}(y_{sjt}) - MC_{sjt})\,y_{sjt}, \tag{61}$$

where the inverse demand follows from substituting $p_j y_{jt} = Y_t$ into (51):

$$p_{sjt} \;=\; \Xi(e_{st})\,y_{jt}^{-\varepsilon}\,y_{sjt}^{\varepsilon-1}\,Y_t, \quad y_{jt} = \left(\Xi(e_{st})y_{sjt}^{\varepsilon} + (1-\Xi(e_{st}))y_{fjt}^{\varepsilon}\right)^{1/\varepsilon}. \tag{62}$$

The Cournot first-order condition $(p - MC)/p = -\partial \ln p/\partial \ln y$ requires the elasticity of inverse demand. Differentiating (62),

$$\frac{\partial \ln p_{sjt}}{\partial \ln y_{sjt}} \;=\; -\varepsilon\,\frac{\partial \ln y_{jt}}{\partial \ln y_{sjt}} + (\varepsilon - 1). \tag{63}$$

The elasticity of y_{jt} with respect to y_{sjt} equals the CES output share, which coincides with the revenue share:

$$\frac{\partial \ln y_{jt}}{\partial \ln y_{sjt}} \;=\; \Xi(e_{st})\left(\frac{y_{sjt}}{y_{jt}}\right)^{\varepsilon} \;=\; \phi_{sjt}. \tag{64}$$

Substituting yields the superstar's pricing condition:

$$\frac{MC_{sjt}}{p_{sjt}} \;=\; 1 + \frac{\partial \ln p_{sjt}}{\partial \ln y_{sjt}} \;=\; \varepsilon(1-\phi_{sjt}). \tag{65}$$

The analogous condition for the aggregate fringe, with own revenue share $1 - \phi_{sjt}$, is

$$\frac{MC_{fjt}}{p_{fjt}} = \varepsilon \phi_{sjt}. \tag{66}$$

Taking the ratio of (66) to (65),

$$\frac{p_{fjt}}{p_{sjt}} = \frac{1 - \phi_{sjt}}{\phi_{sjt}} \frac{MC_{fjt}}{MC_{sjt}}. \tag{67}$$

The inverse demand expressions imply $p_{fjt}/p_{sjt} = \frac{1-\Xi(e_{st})}{\Xi(e_{st})}(y_{fjt}/y_{sjt})^{\varepsilon-1}$. The marginal cost ratio is $MC_{fjt}/MC_{sjt} = \lambda^{m_{jt}}\psi(e_{st}, n_{st})$, which follows from the production functions $y_{sjt} = q_{sjt}\psi l_{sjt}$ and $y_{fjt} = q_{fjt}l_{fjt}$, together with the quality-gap definition $q_{sjt}/q_{fjt} = \lambda^{m_{jt}}$. Substituting yields the relative-output condition:

$$\left(\frac{y_{fjt}}{y_{sjt}}\right)^{\varepsilon-1} = \frac{\Xi(e_{st})}{1 - \Xi(e_{st})} \cdot \frac{1 - \phi_{sjt}}{\phi_{sjt}} \cdot \lambda^{m_{jt}}\, \psi(e_{st}, n_{st}). \tag{68}$$

Equation (68) expresses relative output as an increasing function of the quality gap m_{jt} and the embedded-intangible productivity factor $\psi(e_{st}, n_{st})$: when the superstar's relative productivity rises, its output share rises relative to the fringe.

B.4. Aggregate Output and Growth Rate

Substituting the superstar (6) and fringe-firm (7) output equations into (3) gives

$$\begin{aligned} Y_t &= \exp\left(\int_0^1 \frac{1}{\varepsilon} \ln\left[\Xi(e_{st})\left(q_{sjt}\,\frac{\left((1-\xi)e_{st}\right)^{\alpha}}{\eta\, n_{st}^{\alpha}}\, l_{sjt}\right)^{\varepsilon} + \left(1 - \Xi(e_{st})\right)\left(q_{fjt}\, l_{fst}\right)^{\varepsilon}\right] dj\right) \\ &= Q_t\, \exp\left(\int_0^1 \frac{1}{\varepsilon} \ln\left[\Xi(e_{st})\left(\frac{\left((1-\xi)e_{st}\right)^{\alpha}}{\eta\, n_{st}^{\alpha}}\, l_{sjt}\right)^{\varepsilon} + \left(1 - \Xi(e_{st})\right)\left(\lambda^{-m_{jt}} l_{fst}\right)^{\varepsilon}\right] dj\right), \end{aligned} \tag{69}$$

where $Q_t \equiv \exp\Big(\int_0^1 \ln q_{sjt}\, dj\Big)$ is the geometric mean of superstar quality levels across sectors. Substituting the labor demands from (24) and factoring out ω_t^{-1}:

$$Y_t = Q_t\, \omega_t^{-1} \exp\left(\int_0^1 \frac{1}{\varepsilon} \ln\left[\Xi(e_{st})\left(\frac{\big((1-\xi)e_{st}\big)^{\alpha}}{\eta\, n_{st}^{\alpha}}\frac{\phi_{sjt}}{\sigma_{sjt}}\right)^{\varepsilon} + \big(1-\Xi(e_{st})\big)\Big(\lambda^{-m_{jt}}(1-\phi_{sjt})\Big)^{\varepsilon}\right] dj\right). \tag{70}$$

Since the integrand depends only on the state variables (m,k,n), aggregate output can be written as

$$Y_t = Q_t\, \omega_t^{-1} \exp\left(\sum_{m,k,n} R_t(m,k,n)\, \mu_t(m,k,n)\right), \tag{71}$$

where

$$R_t(m,k,n) \equiv \frac{1}{\varepsilon} \ln\left[\Xi(k)\left(\frac{\big((1-\xi)\theta^{kt}\big)^{\alpha}}{\eta\, n^{\alpha_s}}\frac{\phi_t(m,k,n)}{\sigma_t(m,k,n)}\right)^{\varepsilon} + \big(1-\Xi(k)\big)\Big(\lambda^{-mt}\big(1-\phi_t(m,k,n)\big)\Big)^{\varepsilon}\right]. \tag{72}$$

Taking differences between t and $t+\Delta t$:

$$\begin{aligned}\ln Y_{t+\Delta t} - \ln Y_t = &\big(\ln Q_{t+\Delta t} - \ln Q_t\big) - \big(\ln \omega_{t+\Delta t} - \ln \omega_t\big)\\ &+ \sum_{m,k,n}\Big[R_{t+\Delta t}(m,k,n)\, \mu_{t+\Delta t}(m,k,n) - R_t(m,k,n)\, \mu_t(m,k,n)\Big] + o(\Delta t).\end{aligned}$$

$$\ln Q_{t+\Delta t} - \ln Q_t = \ln\lambda \left[\sum_{m,k,n}\Big(z_t^{\text{Ver}}(m,k,n) + \mathbb{P}_{s\geq s'}\, z_t^{\text{Hor}}(m,k,n) + Z_t^{\text{f}}(m,k,n)\Big)\mu_t(m,k,n)\right]\Delta t + o(\Delta t). \tag{73}$$

Dividing by Δt and letting $\Delta t \to 0$, the growth rate decomposes as

$$g_t = g_{Q,t} - g_{\omega,t} + g_{R,t}, \tag{74}$$

where $g_{Q,t}$ captures quality improvements, $g_{\omega,t}$ real-wage growth, and $g_{R,t}$ changes in the cross-sectional distribution of sectoral states. In steady state, $\mu_t(m,k,n)$ is stationary, so $g_{R,t}=0$; wages grow at the same rate as output, so the real wage is constant and $g_{\omega,t}=0$. The steady-state growth rate is therefore determined solely by quality improvements:

$$g = g_Q = \ln\lambda \sum_{m,k,n} \Big(z^{\text{Ver}}(m,k,n) + \mathbb{P}_{s\geq s'}\, z^{\text{Hor}}(m,k,n) + Z^{\text{f}}(m,k,n) \Big) \mu(m,k,n). \tag{75}$$

B.5. Decomposition of Output

To understand misallocation through markup dispersion across firms, I adapt Peters (2020) and decompose aggregate output as $Y_t = Q_t \times E_t \times M_t \times S_t$, where Q_t captures quality improvements, M_t captures misallocation arising from markup dispersion, and E_t and S_t are residual terms defined below. Starting from (70) and factoring out $\Xi(e_{st})\big(\frac{((1-\xi)e_{st})^{\alpha}}{\gamma n_{st}^{\alpha}}\big)^{\varepsilon}$ from the integrand, aggregate output can be written as

$$Y_t = Q_t\, \omega_t^{-1}\, E_t\, \exp\left(\int_0^1 \frac{1}{\varepsilon} \ln\left[\left(\tfrac{\phi_{sjt}}{\sigma_{sjt}}\right)^{\varepsilon} + \frac{1}{\chi(e_{st})}\left(\frac{\eta\, n_{st}^{\alpha}}{((1-\xi)e_{st})^{\alpha}}\, \lambda^{-m_{jt}}(1-\phi_{sjt}) \right)^{\varepsilon} \right] dj \right), \tag{76}$$

where the factored-out term defines

$$E_t \;=\; \exp\left(\int_0^1 \frac{1}{\varepsilon} \ln\left[\Xi(e_{st}) \left(\frac{((1-\xi)e_{st})^{\alpha}}{\eta\, n_{st}^{\alpha}} \right)^{\varepsilon} \right] dj \right). \tag{77}$$

Multiplying and dividing the integrand in (76) by the linear weight $\frac{\phi_{sjt}}{\sigma_{sjt}} + (1-\phi_{sjt})$ isolates a geometric-mean term and a CES residual:

$$\begin{aligned} Y_t = Q_t\, E_t\, \omega_t^{-1}\, \exp&\left(\int_0^1 \ln\left[\tfrac{\phi_{sjt}}{\sigma_{sjt}} + (1-\phi_{sjt}) \right] dj \right) \\ \times \exp&\left(\int_0^1 \frac{1}{\varepsilon} \ln\left[\frac{\left(\frac{\phi_{sjt}}{\sigma_{sjt}}\right)^{\varepsilon} + \frac{1}{\chi(e_{st})}\left(\frac{\eta\, n_{st}^{\alpha}}{((1-\xi)e_{st})^{\alpha}}\, \lambda^{-m_{jt}}(1-\phi_{sjt}) \right)^{\varepsilon}}{\left(\frac{\phi_{sjt}}{\sigma_{sjt}} + (1-\phi_{sjt}) \right)^{\varepsilon}} \right] dj \right). \end{aligned} \tag{78}$$

Using (34) and defining the multiplicative misallocation term as the ratio of the geometric mean to the arithmetic mean of $\frac{\phi_{sjt}}{\sigma_{sjt}} + (1 - \phi_{sjt})$:

$$M_t = \frac{\exp\left(\int_0^1 \ln \left[\frac{\phi_{sjt}}{\sigma_{sjt}} + (1 - \phi_{sjt}) \right] dj \right)}{\int_0^1 \left[\frac{\phi_{sjt}}{\sigma_{sjt}} + (1 - \phi_{sjt}) \right] dj}, \tag{79}$$

the full output decomposition is[40]

$$Y_t = Q_t \times E_t \times M_t \times S_t, \tag{80}$$

where

$$S_t = \exp\left(\int_0^1 \frac{1}{\varepsilon} \ln \left[\frac{\left(\frac{\phi_{sjt}}{\sigma_{sjt}}\right)^{\varepsilon} + \frac{1}{\chi(e_{st})} \left(\frac{\eta\, n_{st}^{\alpha}}{((1-\xi)e_{st})^{\alpha}} \lambda^{-m_{jt}} (1 - \phi_{sjt}) \right)^{\varepsilon}}{\left(\frac{\phi_{sjt}}{\sigma_{sjt}} + (1 - \phi_{sjt}) \right)^{\varepsilon}} \right] dj \right). \tag{81}$$

By Jensen's inequality, $M_t \leq 1$, with equality only when $\frac{\phi_{sjt}}{\sigma_{sjt}} + (1 - \phi_{sjt})$ is constant across sectors—that is, when markup-weighted revenue shares are equalized.

Table A5 reports M and g under the baseline and two counterfactual scenarios in which scope or embedded barriers are eliminated. The baseline value $M = 0.986$ confirms that markup dispersion generates only a modest static output loss of approximately 1.4 percent. The two counterfactuals, however, reveal a fundamental tension between static efficiency and growth rate.

Table A5. Misallocation and Growth Rate Under Different Entry Barrier Scenarios

	Baseline	No Scope Barrier (n)	No Embedded Barrier (k)
M	0.985	0.994	0.987
g	0.016	0.011	0.024

[40] The term M_t differs slightly from Peters (2020): even with constant markups across superstar firms, dispersion between superstars and fringe firms persists through the market share ϕ_{sjt}.

B.6. Consumption Equivalence Welfare Measure

On the balanced growth path, consumption grows at the steady-state rate g, so $C(t) = C_0\, e^{gt}$, where

$$C_0 = Y_0\left(1 - \int_0^1 \left(I_{0j}^{\mathrm{Ver}} + I_{0j}^{\mathrm{Hor}} + I_{0j}^{\mathrm{Emb}} + I_{0j}^{\mathrm{f}}\right) dj + G_0\right). \tag{82}$$

Here $I_0^{\cdot}$ are the investment-expenditure shares evaluated at the balanced growth path, and G_0 is the lump-sum transfer from the government budget ($G_0 = 0$ in the benchmark; in the policy experiments, G_0 equals the net fiscal surplus rebated to households) given

$$G_0 = \underbrace{\int_0^1 \tau_{\text{scope}}\, \pi_{0j}\, dj}_{\text{Tax revenue}} - \underbrace{\left(\int_0^1 \left(\tau_{\text{ver}} I_{0j}^{\mathrm{Ver}} + \tau_{\text{hor}} I_{0j}^{\mathrm{Hor}} + \tau_{\text{emb}} I_{0j}^{\mathrm{Emb}}\right) dj + \tau_f I_{0j}^{f}\right)}_{\text{Subsidy}}. \tag{83}$$

Welfare is the present discounted value of lifetime utility. Substituting $\ln C(t) = \ln C_0 + gt$ and evaluating the resulting integrals:

$$\Omega = \int_0^\infty e^{-\rho t} \ln C(t)\, \mathrm{d}t = \ln C_0 \int_0^\infty e^{-\rho t}\, \mathrm{d}t \;+\; g \int_0^\infty t\, e^{-\rho t}\, \mathrm{d}t = \frac{1}{\rho}\left(\ln C_0 + \frac{g}{\rho}\right). \tag{84}$$

Consumption-equivalent variation. Let superscripts B and P denote the benchmark and policy economies, respectively, each evaluated on its own balanced growth path. The consumption-equivalent variation δ is the permanent proportional change in benchmark consumption that leaves a household indifferent between the two economies:

$$\Omega^P = \frac{1}{\rho}\left(\ln\left[C_0^B(1+\delta)\right] + \frac{g^B}{\rho}\right). \tag{85}$$

Substituting $\Omega^P = \frac{1}{\rho}(\ln C_0^P + g^P/\rho)$ and solving for δ:

$$\ln\left(\frac{C_0^P}{C_0^B}\right) + \frac{g^P - g^B}{\rho} = \ln(1+\delta), \tag{86}$$

$$\delta = \frac{C_0^P}{C_0^B} \exp\left(\frac{g^P - g^B}{\rho}\right) - 1. \tag{87}$$

The measure $\Delta\Omega\,(\%) \equiv 100\,\delta$ is reported in Figures 9 and C4. A value $\delta > 0$ means the policy economy delivers higher welfare: a benchmark household requires δ additional lifetime consumption to be indifferent between the benchmark and the policy allocation. A value $\delta < 0$ means the benchmark is preferred: households require compensation to accept the policy.

B.7. The Social Planner's Problem

The social planner solves

$$\max_{\left[\{l_{sjt}, l_{fjt}, z_{jt}^{\text{Ver}}, z_{jt}^{\text{Hor}}, z_{jt}^{\text{Emb}}, z_{jt}^{f}\}_{j\in[0,1]}\right]} \int_0^{+\infty} e^{-\rho t} \ln(C_t)\, dt \tag{88}$$

subject to the resource constraint, the cost functions for each innovation margin, the production technology, and the law of motion for the firm distribution:

$$C_t + \int_0^1 \left(I_{jt}^{\text{Ver}} + I_{jt}^{\text{Hor}} + I_{jt}^{\text{Emb}} + I_{jt}^{f} \right) dj \le Y_t, \tag{89}$$

$$I_{jt}^{x} = \gamma^{x} \left(z_{jt}^{x} \right)^{\vartheta x} Y_t, \qquad x \in \{\text{Ver}, \text{Hor}, \text{Emb}, f\}, \tag{90}$$

$$\ln(Y_t) = \int_0^1 \ln(y_{jt})\, dj, \qquad y_{jt} = \left[\Xi(e_{st})\, y_{sjt}^{\varepsilon} + \left(1 - \Xi(e_{st})\right) y_{fjt}^{\varepsilon} \right]^{1/\varepsilon}, \tag{91}$$

$$y_{sjt} = q_{sjt}\, \psi(e_{st}, n_{st})\, l_{sjt}, \qquad y_{fjt} = q_{fjt}\, l_{fjt}, \qquad \psi(e_{st}, n_{st}) = \frac{\left((1-\xi)\, e_{st}\right)^{\alpha}}{\eta\, n_{st}^{\alpha}}, \tag{92}$$

$$\int_0^1 \left(l_{sjt} + l_{fjt} \right) dj \le 1, \qquad q_{sjt} = \lambda^{m_{jt}}\, q_{sj0}, \qquad e_{st} = \theta^{k_{st}}\, e_{s0}, \tag{93}$$

$$\begin{aligned} \dot{\mu}_t(m,k,n) = {} & z_t^{\text{Ver}}(m-1,k,n)\, \mu_t(m-1,k,n) + z_t^{\text{Emb}}(m,k-1,n)\, \mu_t(m,k-1,n) \\ & + \mathbb{P}_{s>s'}\, z_t^{\text{Hor}}(m,k,n-1)\, \mu_t(m,k,n-1) \\ & - \Big[z_t^{\text{Ver}}(m,k,n) + z_t^{\text{Emb}}(m,k,n) + \mathbb{P}_{s>s'}\, z_t^{\text{Hor}}(m,k,n) \\ & \quad + \mathbb{I}_{f>s}\, Z_t^{f}(m,k,n) + Z_{c>s,t}^{\text{Hor}} \Big]\, \mu_t(m,k,n). \end{aligned} \tag{94}$$

B.7.1. Static Output Maximization

For a given distribution $\mu_t(m,k,n)$ and a given set of innovation rates, the social planner maximizes aggregate output by choosing labor allocations $\{l_{sjt}, l_{fjt}\}_{j\in[0,1]}$ subject to the aggregate labor feasibility constraint. The first-order conditions equate the marginal social product of labor across all firms within and across sectors, yielding the optimal intrasectoral labor ratio:

$$\frac{l_{sjt}^{SP}}{l_{fjt}^{SP}} = \left(\frac{\Xi(e_{st})}{1-\Xi(e_{st})}\right)^{\frac{1}{1-\varepsilon}} \left(\frac{q_{sjt}\,\psi_{st}}{q_{fjt}}\right)^{\frac{\varepsilon}{1-\varepsilon}}, \tag{95}$$

where $\psi_{st} \equiv \psi(e_{st}, n_{st})$. Equation (95) together with the aggregate labor constraint pins down $\{l_{sjt}^{SP}, l_{fjt}^{SP}\}$ as functions of the state (m,k,n) and the firm distribution.

Comparison with the decentralized equilibrium. In the decentralized equilibrium, superstar firms charge a markup $\sigma_{st} \equiv \sigma(m,k,n) > 1$ over their marginal cost, which inflates the price of superstar goods relative to fringe goods within each sector. The final-goods producer's cost-minimization problem then yields a decentralized labor ratio that differs from (95) by the markup factor:

$$\frac{l_{sjt}^{DE}}{l_{fjt}^{DE}} = \frac{1}{\sigma_{st}^{1/(1-\varepsilon)}} \cdot \frac{l_{sjt}^{SP}}{l_{fjt}^{SP}}. \tag{96}$$

Since $\sigma_{st} > 1$ and $\varepsilon < 1$, the multiplicative factor is strictly less than one: superstar production is underemployed relative to the social optimum, and labor is correspondingly overallocated to the fringe within each sector. This static wedge is the standard distortion from imperfect competition and is uniform across firm states up to the heterogeneity in markups themselves. Section B.5 discusses the misallocation arising from markups and its relationship with the growth rate. The dynamic inefficiencies of the model, by contrast, arise from externalities embedded in firms' innovation choices and *cannot* be corrected by uniform interventions. The remainder of this appendix characterizes these dynamic externalities through the planner's Hamilton-Jacobi-Bellman equation and derives the corresponding Pigouvian instruments.

B.7.2. Dynamic Innovation Problem

Let $v^{SP}(m,k,n)$ denote the normalized social value of a sector occupied by a superstar in state (m,k,n) along the balanced growth path. Three classes of externalities distinguish the planner's problem from the decentralized firm's problem and require explicit correction in the HJB equation: (i) a *scope-barrier externality,* by which expansion to an additional sector raises entry barriers across all sectors the firm currently occupies; (ii) an *embedded-barrier externality,* by which embedded-intangible accumulation raises entry barriers across all currently active sectors; and (iii) a *business-stealing externality,* by which creative destruction reallocates social surplus between incumbents and challengers. The two barrier externalities require explicit correction terms, denoted $\Psi^{\text{Scope}}(k,n)$ and $\Psi^{\text{Emb}}(k,n)$, derived below.

The planner's normalized HJB equation along the BGP is

$$\begin{aligned}
\rho\, v^{SP}(m,k,n) = \max_{z^{\text{Ver}},\, z^{\text{Hor}},\, z^{\text{Emb}},\, z^{f}} \Big\{ & \Phi^{SP}(m,k,n) \\
& + z^{\text{Ver}}\big[v^{SP}(m+1,k,n) - v^{SP}(m,k,n)\big] \\
& + \mathbb{P}_{s>s'}(k,n)\, z^{\text{Hor}}\big[\tilde{v}^{SP}(m,k,n+1) - v^{SP}(m,k,n) - \Psi^{\text{Scope}}(k,n)\big] \\
& + z^{\text{Emb}}\big[v^{SP}(m,k+1,n) - v^{SP}(m,k,n) - \Psi^{\text{Emb}}(k,n)\big] \\
& + Z^{\text{Hor}}_{c>s}(k,n)\big[\bar{v}^{SP}(1,k,n) - v^{SP}(m,k,n)\big] \\
& + \mathbb{I}_{f>s}(k,n)\, z^{f}(m,k,n)\big[v^{SP}(\bar{m},1,1) - v^{SP}(m,k,n)\big] \\
& - \sum_{x\in\{\text{Ver},\text{Hor},\text{Emb},f\}} \gamma^{x}(z^{x})^{\vartheta^{x}} \Big\},
\end{aligned} \tag{97}$$

where $\Phi^{SP}(m,k,n) \equiv \ln \tilde{y}_j(m,k,n)$ is the planner's per-sector normalized log-output evaluated at the optimal allocation in (95) and $\tilde{v}^{SP}(m,k,n{+}1)$ is the social value after the firm acquires one additional sector with quality gap reset to one. The first three terms inside the maximum capture vertical, horizontal, and embedded innovation by the incumbent; the remaining terms capture creative destruction by challenger superstars and by the fringe; the last term aggregates innovation costs across all three margins.

Welfare gain from creative destruction. Let $W^{\Delta}(k,n)$ denote the expected social welfare gain per creative-destruction event in a sector occupied by an incumbent with embedded state k and scope n:

$$W^{\Delta}(k,n) \;\equiv\; \mathbb{E}\big[v^{SP}(\bar{m},1,1) - v^{SP}(m,k,n) \;\big|\; \text{displacement in sector with incumbent state } (k,n)\big] > 0. \tag{98}$$

Because a creative-destruction event resets the affected sector to a high-productivity entrant state $(\bar{m},1,1)$, $W^{\Delta}(k,n)$ is strictly positive: each deterred entry represents foregone social welfare.

Reductions in entry probability. Define the reduction in the probability of any challenger displacing the incumbent when the incumbent's scope rises from n to $n+1$:

$$\begin{aligned}\mathcal{D}^{\text{Scope}}(k,n) \;\equiv\; &\sum_{m_c,k_c,n_c} \big[\mathbb{I}_{c>s}(k_c,n_c;\,k,n) - \mathbb{I}_{c>s}(k_c,n_c;\,k,n+1)\big]\, z^{\text{Hor}}(m_c,k_c,n_c)\,\mu(m_c,k_c,n_c)\\ &+ \big[\mathbb{I}_{f>s}(k,n) - \mathbb{I}_{f>s}(k,n+1)\big]\, Z^{f}(m,k,n),\end{aligned} \tag{99}$$

and the analogous reduction when embedded state rises from k to $k+1$:

$$\begin{aligned}\mathcal{D}^{\text{Emb}}(k,n) \;\equiv\; &\sum_{m_c,k_c,n_c} \big[\mathbb{I}_{c>s}(k_c,n_c;\,k,n) - \mathbb{I}_{c>s}(k_c,n_c;\,k+1,n)\big]\, z^{\text{Hor}}(m_c,k_c,n_c)\,\mu(m_c,k_c,n_c)\\ &+ \big[\mathbb{I}_{f>s}(k,n) - \mathbb{I}_{f>s}(k+1,n)\big]\, Z^{f}(m,k,n).\end{aligned} \tag{100}$$

Both operators are non-negative because $\mathbb{I}_{c>s}(k_c,n_c;\,k,n)$ is weakly decreasing in the incumbent's k and n: a stronger incumbent is harder to displace.

Scope-barrier externality. When the incumbent expands from n to $n+1$ sectors, the resulting increase in scope raises the entry barrier across all $n+1$ sectors the firm occupies after the expansion. The present discounted social cost of this barrier reinforcement is

$$\Psi^{\text{Scope}}(k,n) \;\equiv\; \frac{n+1}{\rho}\,\mathcal{D}^{\text{Scope}}(k,n)\cdot W^{\Delta}(k,n) \;\geq\; 0. \tag{101}$$

The factor $n+1$ reflects that the barrier increase operates across all sectors the firm occupies after the expansion. Ψ^{Scope} is strictly increasing in n: wider-scope incumbents generate disproportionately large scope-barrier externalities per expansion step.

Embedded-barrier externality. Raising k by one step increases the firm-level embedded-intangible stock, raising the entry barrier across all n currently active sectors. The per-step present discounted social cost is

$$\Psi^{\text{Emb}}(k,n) \;\equiv\; \frac{n}{\rho}\,\mathcal{D}^{\text{Emb}}(k,n)\cdot W^{\Delta}(k,n) \;\geq\; 0, \tag{102}$$

where the factor n reflects the firm's currently active sectors at the time of investment. Ψ^{Emb} is also strictly increasing in n: firms with larger scope generate disproportionately large embedded-barrier externalities per investment dollar.

B.8. Optimal Pigouvian Instruments

For each innovation margin $j \in \{\text{Ver}, \text{Hor}, \text{Emb}, f\}$, a multiplicative cost instrument τ^j enters the firm's cost as $(1-\tau^j)\gamma^j(z^j)^{\vartheta^j}$. Following the convention used throughout the main text, $\tau^j > 0$ denotes a *subsidy* (cost reduced) and $\tau^j < 0$ denotes a *tax* (cost raised). The instrument is calibrated so that the firm's modified first-order condition replicates the planner's. Equating the two FOCs gives the general formula

$$\tau^{j*} \;=\; 1 \;-\; \frac{p_j}{s_j}, \tag{103}$$

where p_j is the firm's private marginal return from innovation on margin j at the decentralized equilibrium and s_j is the planner's social marginal return at the social optimum. When $s_j > p_j$ (margin under-supplied), $\tau^{j*} > 0$ and the instrument is a subsidy; when

$s_j < p_j$ (margin over-supplied), $\tau^{j*} < 0$ and the instrument is a tax.

B.8.1. Vertical Innovation

Define the vertical consumer-surplus wedge

$$\Delta CS^{\text{Ver}}(m,k,n) \;\equiv\; \big[v^{SP}(m+1,k,n) - v^{SP}(m,k,n)\big] - \big[v(m+1,k,n) - v(m,k,n)\big] \;\geq\; 0, \tag{104}$$

which captures the consumer-surplus gain from a vertical step that the firm does not internalize. The inequality is strict whenever the planner's markup-corrected surplus share exceeds the firm's profit margin. Applying (103),

$$\tau^{\text{Ver}*}(m,k,n) \;=\; \frac{\Delta CS^{\text{Ver}}(m,k,n)}{v^{SP}(m+1,k,n) - v^{SP}(m,k,n)} \;\geq\; 0. \tag{105}$$

Because the social return strictly exceeds the private return and both are positive, $\tau^{\text{Ver}*} \geq 0$ uniformly: vertical innovation is always under-supplied and calls unambiguously for a subsidy.

B.8.2. Horizontal Innovation

Horizontal innovation involves two competing externalities relative to the private FOC: a consumer-surplus wedge $\Delta CS^{\text{Hor}}(m,k,n) \geq 0$ that calls for a subsidy, and the scope-barrier penalty $\Psi^{\text{Scope}}(k,n) \geq 0$ that calls for a tax. Applying (103),

$$\tau^{\text{Hor}*}(m,k,n) \;=\; 1 \;-\; \frac{\tilde{v}(m,k,n+1) - v(m,k,n)}{\tilde{v}^{SP}(m,k,n+1) - v^{SP}(m,k,n) - \Psi^{\text{Scope}}(k,n)}. \tag{106}$$

The sign of $\tau^{\text{Hor}*}$ depends on scope. For small n, ΔCS^{Hor} dominates Ψ^{Scope} and the instrument is a subsidy ($\tau^{\text{Hor}*} > 0$); for large n, $\Psi^{\text{Scope}}(k,n)$ dominates and the instrument is a tax ($\tau^{\text{Hor}*} < 0$). The threshold $n^{\dagger}$ at which the instrument switches sign satisfies

$$\Delta CS^{\text{Hor}}(m,k,n^{\dagger}) \;=\; \Psi^{\text{Scope}}(k,n^{\dagger}). \tag{107}$$

B.8.3. Embedded Intangible Investment

Embedded intangibles play a structurally distinct role because of their *dual function*: they simultaneously raise the quality of the superstar's products (through both brand value ξe and organizational capital $(1-\xi)e$) and raise the entry barrier that challengers must overcome. Define the embedded consumer-surplus wedge

$$\Delta CS^{\text{Emb}}(m,k,n) \;\equiv\; \big[v^{SP}(m,k+1,n) - v^{SP}(m,k,n)\big] - \big[v(m,k+1,n) - v(m,k,n)\big] \;\geq\; 0. \tag{108}$$

Applying (103),

$$\tau^{\text{Emb}*}(m,k,n) \;=\; 1 \;-\; \frac{v(m,k+1,n) - v(m,k,n)}{v^{SP}(m,k+1,n) - v^{SP}(m,k,n) - \Psi^{\text{Emb}}(k,n)}. \tag{109}$$

The sign is determined by the balance between ΔCS^{Emb} and Ψ^{Emb}:

- If $\Delta CS^{\text{Emb}}(m,k,n) > \Psi^{\text{Emb}}(k,n)$, the consumer-surplus gain dominates and $\tau^{\text{Emb}*} > 0$ (subsidy).
- If $\Psi^{\text{Emb}}(k,n) > \Delta CS^{\text{Emb}}(m,k,n)$, the barrier externality dominates and $\tau^{\text{Emb}*} < 0$ (tax).

Because $\Psi^{\text{Emb}}(k,n)$ is strictly increasing in both k and n, the instrument switches from subsidy to tax as the incumbent accumulates embedded capital and expands scope.

B.8.4. Fringe Entry

The fringe consumer-surplus wedge is

$$\Delta CS^{f}(m,k,n) \;\equiv\; \big[v^{SP}(\bar{m},1,1) - v^{SP}(m,k,n)\big] - \big[v(\bar{m},1,1) - v_f(m,k,n)\big] \;\geq\; 0. \tag{110}$$

Applying (103),

$$\tau^{f*}(m,k,n) \;=\; 1 \;-\; \frac{\mathbb{I}_{f>s}(k,n)\big[v(\bar{m},1,1) - v_f(m,k,n)\big]}{\mathbb{I}_{f>s}(k,n)\big[v^{SP}(\bar{m},1,1) - v^{SP}(m,k,n)\big]}. \tag{111}$$

To sign τ^{f*}, decompose the social value of a sector as $v^{SP}(s) = v(s)+v_f(s)+CS(s)$, where $CS(s)$ is consumer surplus. Substituting into $\tau^{f*} > 0$ yields the condition

$$CS(\bar{m}, 1, 1) \ + \ v_f(\bar{m}, 1, 1) \ > \ v(m, k, n) \ + \ CS(m, k, n). \tag{112}$$

The left-hand side collects the post-entry consumer surplus and the fringe option value in the new state; the right-hand side collects the value of the destroyed incumbent and the pre-entry consumer surplus. Whether the instrument is a subsidy or a tax therefore depends on the quantitative balance between the consumer-surplus and option-value gains of creative destruction and the business-stealing loss.

B.9. From First-Best to Feasible Instruments

The state-contingent instruments $\{\tau^{\text{Ver}*}, \tau^{\text{Hor}*}, \tau^{\text{Emb}*}, \tau^{f*}\}$ constitute the first-best Pigouvian policy but are infeasible: implementation requires conditioning on the unobservable quality-gap vector $\mathbf{m}$ and the embedded-capital state k. A realistic fiscal authority is restricted to taxing operational profits and subsidizing investment costs, and a flat profit tax compresses every continuation-value difference $v(m', k', n') - v(m, k, n)$ proportionally, attenuating vertical, horizontal, embedded, and fringe innovation simultaneously.

The Pigouvian analysis nevertheless delivers signed restrictions that discipline a feasible design. The vertical correction $\tau^{\text{Ver}*} \geq 0$ holds uniformly, since the quality-ladder spillover is unambiguously positive. The horizontal and embedded corrections are unambiguously negative for broad-scope incumbents, since scope expansion and embedded-capital accumulation jointly raise the entry barriers that suppress creative destruction. The fringe correction is sign-ambiguous: equation (112) shows that τ^{f*} is positive when the consumer-surplus and option-value gains of creative destruction dominate the business-stealing loss, and negative when the reverse holds, so the prescription can flip from a pure subsidy to a pure tax under otherwise identical model structure.

These restrictions motivate the second-best instrument set $\{\tau_{\text{scope}}(n > \tau^*),\ \tau_{\text{ver}}(n \leq \tau^*),\ \tau_f\}$ examined in Section 6. The scope-conditional profit tax targets the negative horizontal and embedded wedges jointly, exploiting the regularity that broad-scope incumbents accumulate embedded intangibles most aggressively. The vertical subsidy restricted to $n \leq \tau^*$ implements the uniformly positive Pigouvian correction without subsidizing barrier-building. The fringe instrument τ_f implements the calibration-specific finding $\tau^{f*} > 0$; under parameterizations in which the business-stealing margin dominates, the same Pigouvian logic would prescribe a fringe tax. Three of the four instrument signs—a subsidy on vertical innovation and taxes on horizontal and embedded margins—are pinned down by the model itself; only the sign of τ_f depends on the calibrated parameters.

C. Numerical Appendix

C.1. Additional Numerical Results

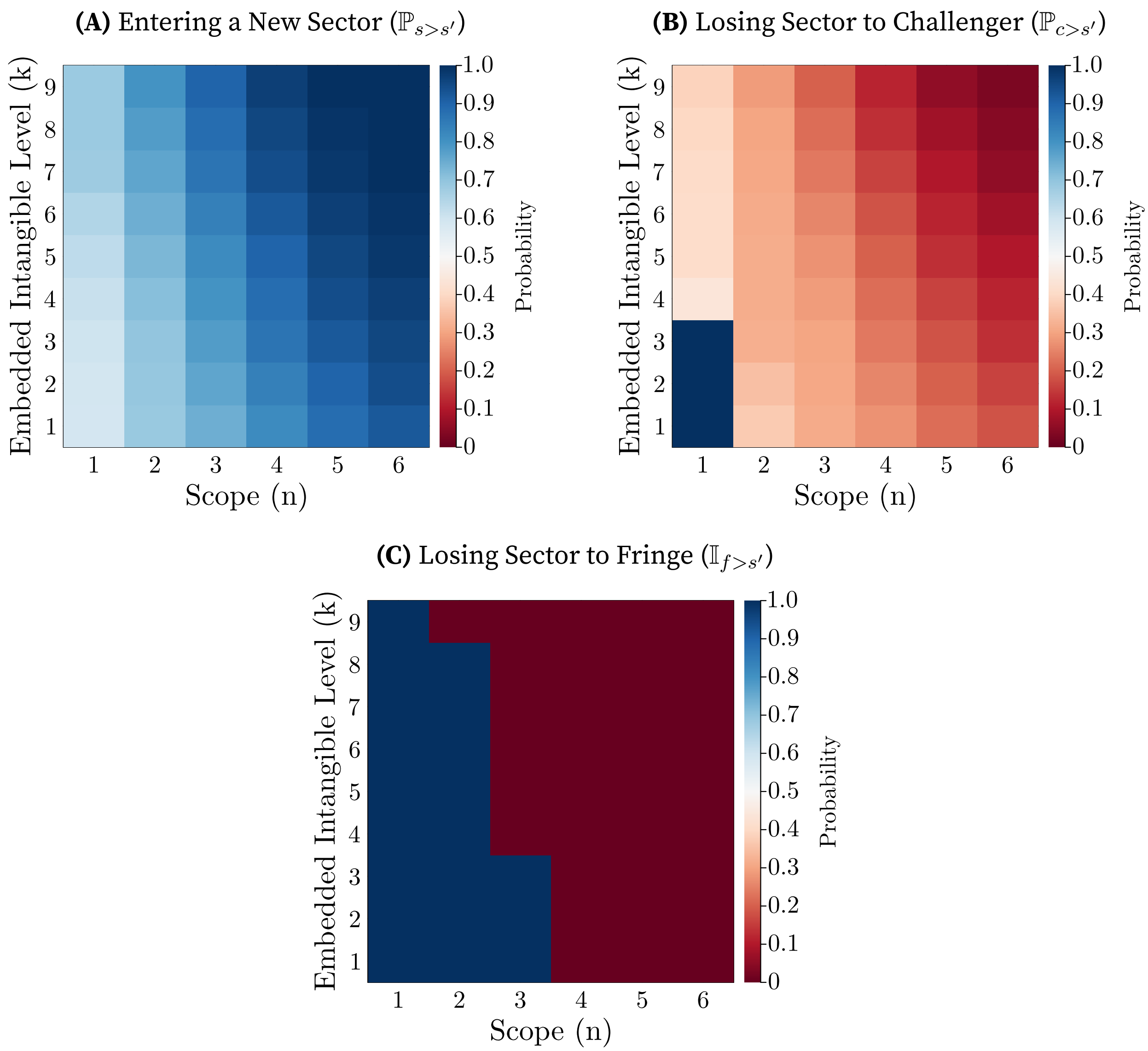


Figure C1. Superstar Firm Entry and Displacement Probabilities

Note: Each panel displays a probability evaluated over the (n, k) grid, where the horizontal axis reports firm scope n and the vertical axis reports the embedded intangible level k. Darker blue indicates higher probability and darker red indicates lower probability.

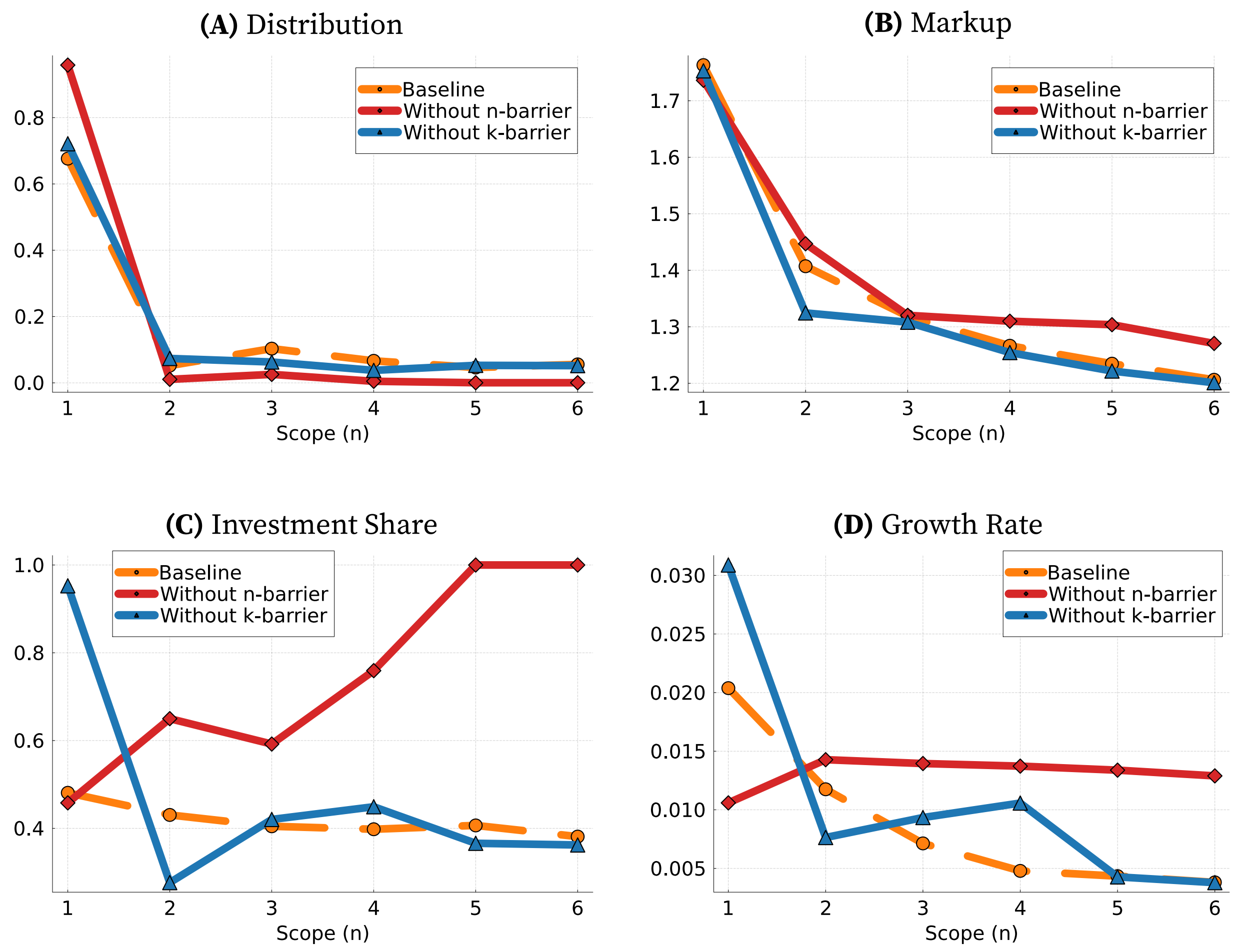


Figure C2. Removing Entry Barrier Scenarios

Note: The figure reports model outcomes as a function of scope n under the baseline specification and two counterfactual experiments: (i) removing n-barrier and (ii) removing k-barrier. In all panels, the dashed orange line denotes the baseline, the solid red line without n-barrier, and the solid blue line without k-barrier.

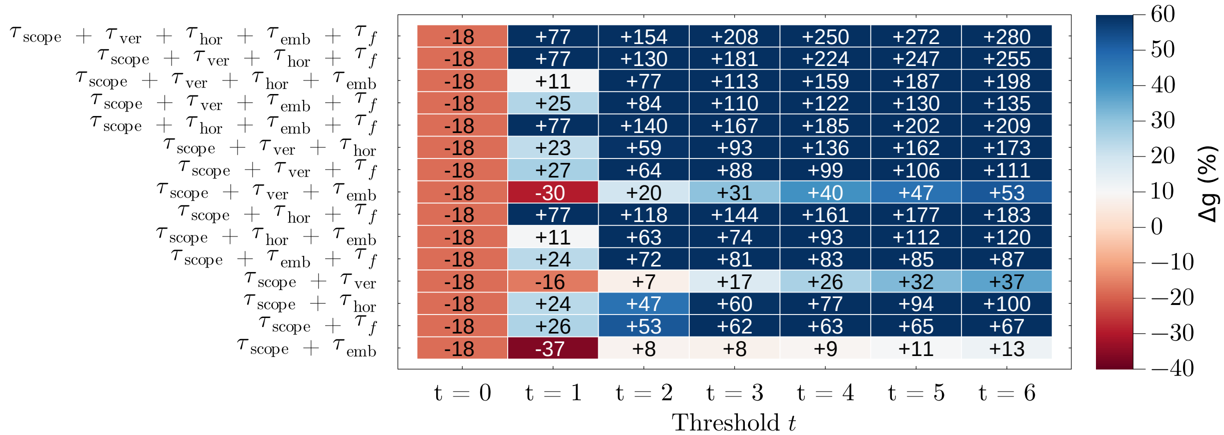


Figure C3. Progressive Policies: Effect on the Growth Rate

Note: Each cell reports the percentage change in the aggregate balanced-growth-path rate Δg (%) relative to the baseline. Rows index the subsidy instrument(s) directed at firms with $n \leq t$; columns index the threshold t. For a given t, superstar firms with $n > t$ face the scope profit tax $\tau_{\text{scope}} = 0.10$, while firms with $n \leq t$ receive cost reductions of magnitude 0.50 on the indicated investment margins, each entering as $(1 - \tau)$ on the respective cost; τ_f applies to fringe entrants at all scope levels. Blue (red) cells denote growth gains (losses); the color scale is linear over $[-40\%, +60\%]$.

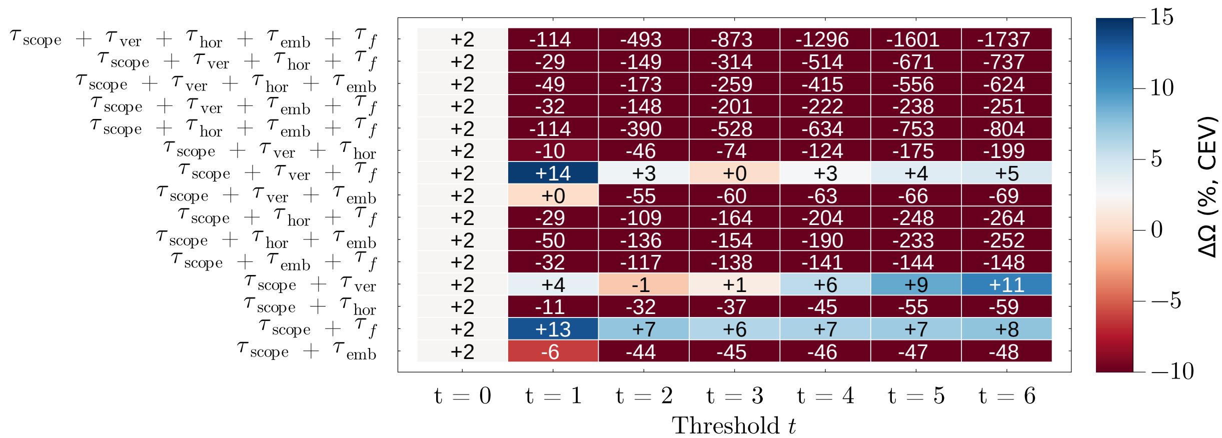
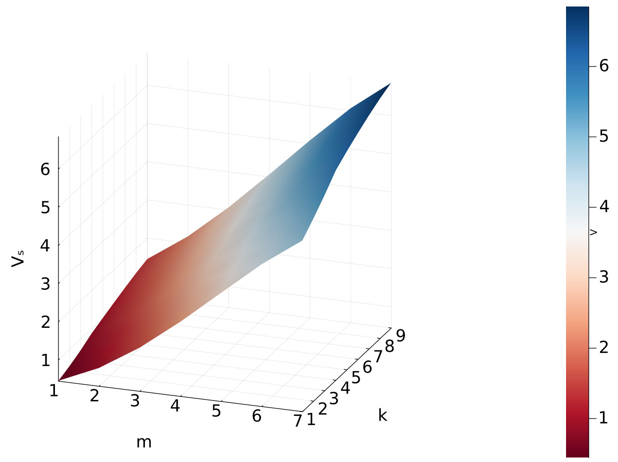


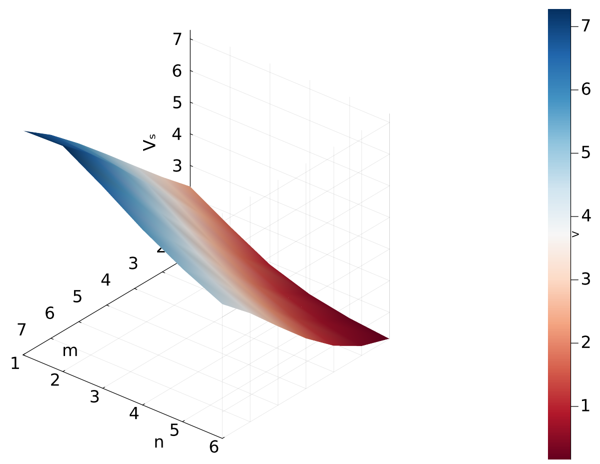
Figure C4. Progressive Policies: Effect on Welfare

Note: Each cell reports the consumption-equivalent welfare gain $\Delta\Omega$ (%, CEV) relative to the baseline. Rows index the subsidy instrument(s) directed at firms with $n \leq t$; columns index the threshold t. For a given t, superstar firms with $n > t$ face the scope profit tax $\tau_{\text{scope}} = 0.10$, while firms with $n \leq t$ receive cost reductions of magnitude 0.50 on the indicated investment margins, each entering as $(1 - \tau)$ on the respective cost; τ_f applies to fringe entrants at all scope levels. Blue (red) cells denote welfare gains (losses); the color scale is linear over $[-10\%, +15\%]$ and saturates below -10%, so the deep-red block in the upper-left reports values ranging from -29% to -1737% (see numerical entries). See Appendix B.6 for the welfare accounting procedure.

(A) $V_s(m, k)$



(B) $V_s(m, n)$



(C) $V_s(k, n)$

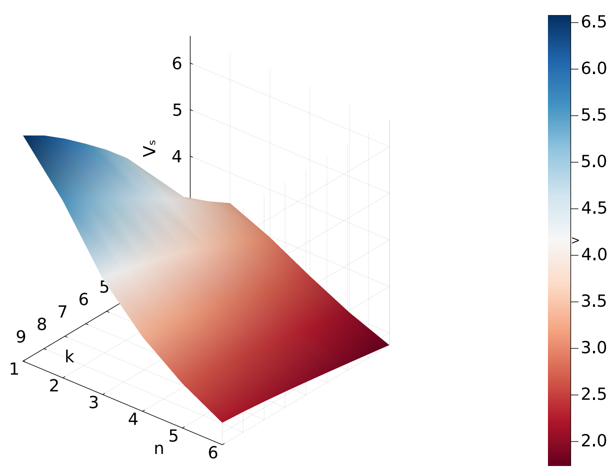


Figure C5. Superstar Value Function

Table C1. Sensitivity Analysis: Effect of a $\pm 1\%$ Parameter Perturbation on Outcomes

	Investment Ratio		Markup		Growth Rate	
Parameter	$+1\%$	-1%	$+1\%$	-1%	$+1\%$	-1%
Demand and Technology						
CES parameter (ε)	-0.372	1.191	-0.244	0.374	0.369	0.628
Curvature of managerial quality (α)	-0.997	0.727	0.088	0.096	-0.271	0.870
Curvature of brand value (β)	-0.039	0.039	-0.025	0.025	-0.053	0.054
Scale of managerial quality (η)	-1.574	2.082	-0.686	0.735	-4.102	4.692
Share of brand value (ξ)	-0.150	0.149	-0.064	0.064	-0.390	0.391
Quality improvement step size (λ)	14.478	-8.754	6.102	-4.538	37.892	-27.825
Embedded innovation step size (θ)	4.840	-13.501	0.941	1.404	15.942	-21.196
Curvature and Cost Scale Parameters						
Curvature of embedded innovation (ϑ_{Emb})	0.896	-0.910	0.010	-0.011	0.603	-0.612
Cost scale of internal innovation (γ^{Ver})	-0.180	0.182	0.000	0.000	-0.265	0.272
Cost scale of horizontal innovation (γ^{Hor})	-0.328	0.337	-0.001	0.002	-0.810	0.836
Cost scale of embedded innovation (γ^{Emb})	0.218	-0.220	-0.006	0.006	-0.198	0.200
Cost scale of fringe (γ^{f})	-0.236	0.239	-0.029	0.029	-0.850	0.864
Baseline value	0.4560		1.6103		0.0159	

Notes: Each entry reports the percentage change in the outcome when the row parameter is perturbed by $+1\%$ or -1%, holding all other parameters at their baseline values. Asymmetry between the $+1\%$ and -1% columns reflects local nonlinearity of the equilibrium mapping.

C.2. Solution Algorithm

This section characterizes the numerical algorithm for computing the balanced growth path equilibrium over the three-dimensional state space (m, k, n). The solution involves finding the superstar value functions $v(m, k, n; \mu)$ and fringe value functions $v_f(m, k, n; \mu)$, the innovation rates z^{Ver}, z^{Emb}, z^{Hor}, z^{f}, and the stationary distribution $\mu(m, k, n)$ that jointly satisfy the model's equilibrium conditions. A key feature of the model is that the value functions (equations (26)–(27)) depend directly on the distribution μ (equation (32)), which in turn is determined by the innovation rates defined in equations (28)–(31) that are themselves derived from the value functions.

BGP Equilibrium Solution:

1. **Compute static values:** Calculate static market shares and profit values using equations (19) and (23).

2. **Initialization:** Initialize the value functions $v^{(0)}(m, k, n; \mu^{(0)})$ and $v_f^{(0)}(m, k, n; \mu^{(0)})$, and the stationary distribution $\mu^{(0)}(m, k, n)$.

3. **Outer Loop: Iterate over (v, μ) until joint convergence**

   Repeat until $\max |v^{\text{new}} - v^{\text{old}}| < \varepsilon$ and $\max |\mu^{\text{new}} - \mu^{\text{old}}| < \varepsilon$:

   (a) **Step 1: Solve HJB Equations (Inner Loop, given μ^{old})**

   - Set $v^{\text{old}}(m, k, n; \mu^{\text{old}})$.
   - Repeat until $\max |v^{\text{new}} - v^{\text{old}}| < \varepsilon$:
     - Compute policy functions $z^{\text{Ver}}, z^{\text{Emb}}, z^{\text{Hor}}, z^f$ from the first-order conditions using v^{old}.
     - Solve the discretized HJB equations for $v^{\text{new}}(m, k, n; \mu^{\text{old}})$.
     - Update $v^{\text{old}} \leftarrow v^{\text{new}}$.

   (b) **Step 2: Solve the Kolmogorov Forward Equation (Inner Loop, given v^{new})**

   - Set $\mu^{\text{old}}(m, k, n; v^{\text{new}})$.
   - Repeat until $\max |\mu^{\text{new}} - \mu^{\text{old}}| < \varepsilon$:
     - Solve the KFE for $\mu^{\text{new}}(m, k, n; v^{\text{new}})$ using the policy functions implied by v^{new}.
     - Update $\mu^{\text{old}} \leftarrow \mu^{\text{new}}$.

   (c) **Step 3: Update and check outer convergence**

   Pass μ^{new} back into the HJB equations, updating $v^{\text{old}} \leftarrow v^{\text{new}}$ and $\mu^{\text{old}} \leftarrow \mu^{\text{new}}$, and repeat Steps 1–2 until both v^{new} and μ^{new} have converged jointly.

   (d) **Parameter Estimation:** To find parameters, search over the parameter space to minimize the objective function

$$\mathcal{L} = \sum_{z=1}^{Z} \frac{\big|\text{model}(z) - \text{data}(z)\big|}{\frac{1}{2}\big|\text{model}(z)\big| + \frac{1}{2}\big|\text{data}(z)\big|}.$$